arXiv:0706.0237v1 [quant-ph] 1 Jun 2007

Quantum Mechanics in Phase Space.

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June 1, 2007

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Abstract

In this paper, I am presenting the basics of the Wigner formulation of Quantum-Mechanics in a very simple language and write about few interpretational stuff raised around it.

1 Introduction

In classical mechanics, a system is completely specified by the position and momentum of its particles. It is common to build a $6N$ dimensional space, where N is the number of particle constituting the system. This space is called the phase space and contains an axe for each of the coordinate and momentum of the system. The system is completely specified by a point in this space. Usually, when we have a lack of knowledge regarding the state of the system, we try to find a probability distribution for the system in its phase space. This is the case in classical statistical mechanics, where $f(p,q)$ (p is a $3N$ dimensional vector representing the momentum of all particles, and q is a $3N$ dimensional vector representing the coordinate of all particles) is the probability of finding the system with coordinate between q and $q+dq$ and momentum between p and $p+dp$. The predictions of quantum mechanics are similar to that of classical statistical mechanics in the sense that they are statistical in nature. Thus, the fundamental question arise whether quantum processes can be described as an average over uniquely determined processes or not. And the observability of these processes if the answer is yes [1]. Naturally the place to look for such processes is the phase space and if the system is undergoing a process in the phase space it must be represented by a point in the phase space. Regardless of the tough problem mentioned, it seems natural for a scientist to seek a function similar to distribution functions in quantum mechanics. But, in the framework of Orthodox and Copenhagen interpretations of quantum mechanics this is impossible.

2 Wigner distribution function

The first example of such a function in quantum mechanics was suggested by Wigner [2]. He mentioned the canonical ensemble probability distribution in classical statistical mechanics for the system having momentum between p and $p+dp$ and coordinate between q and $q+dq$, which is $e^{-\beta\epsilon}$, where β is the reciprocal of the temperate, T, and ϵ is the sum of the kinetic and the potential energy. He mentioned that in quantum mechanics, we cannot simultaneously have momentum and position so we cannot have such an expression. But even if we consider the coordinate alone, where the classical expression for probability is $e^{-\beta V}$, where V is the potential energy of the system, the classical expression is not valid for quantum systems, because when $\beta \to \infty$ there is no reason for that expression to be equal to $|\psi_0(x_1, ..., x_n)|^2$ (the ground state wave function is not even always known). Also, the statistical mechanics of quantum systems is given by the von Newman formula, i.e., $< Q >= Tr(Qe^{-\beta H})$, where Q is the operator corresponding to the quantity under consideration, H is the Hamiltonian of the system and $<>$ denote the expectation value. Because it was not easy to use the von Newman formula for evaluating the expectation values, Wigner suggests to build the following expression

$$P(q,p) = (\frac{1}{\hbar\pi})^n \int dy \psi(q+y/2)^* \psi(q-y/2) e^{ipy/\hbar}, \tag{1}$$

and call it the probability function, here n is the number of dimensions of the space. Thorough out this paper consider the limits of the integrals from $-\infty$ to ∞, unless otherwise is explicitly stated. Unfortunately, Wigner never mentioned how he made up this recipe, he just mention in his first paper regarding distribution functions that: "This expression was found by L. Szilard and the present author some years ago for another purpose." By introducing the inverse Fourier transform of ψ, i.e., $\psi(q) = (\frac{1}{2\pi\hbar})^{\frac{n}{2}} \int dp e^{\frac{i}{\hbar}pq} \psi(p)$ in the above relation we could get

$$\int dy \int dp' \int dp'' \psi(p')^* \psi(p'') e^{\frac{i}{\hbar}[-p'(q+y/2)+p''(q-y/2)+py]}, \tag{2}$$

by performing the integral over y we could get $(2\pi\hbar)^n \delta(p - \frac{p''+p'}{2})$, then we can perform the integral over p'' and perform the change of variables $p-p' \rightarrow -y/2$to get

$$P(q,p) = (\frac{1}{\hbar\pi})^n \int dy \psi(p+y/2)^* \psi(p-y/2) e^{iqy/\hbar}. \tag{3}$$

This relation is completely equivalent with the relation (1) and shows the symmetry of the Wigner functions with respect to q and p. The phase space function corresponding to an operator A is defined thorough

$$A(q,p) = \int dy e^{ipy/\hbar} < q - \frac{y}{2} | \hat{A} | q + \frac{y}{2} > . \tag{4}$$

3 Proposals for getting the Wigner function

I have run to few ways to get the Wigner function in the literature and am going to present these methods here. Stenholm presents a derivation for the Wigner function [3]. All the information extractable from the quantum theory is contained in the matrix elements

$$< x_1 | \hat{\rho} | x_2 >= \psi(x_1) \psi(x_2)^*. \tag{5}$$

We can bring the density matrix into momentum representation and write

$$< p_1 | \hat{\rho} | p_2 >= \frac{1}{2\pi\hbar} \int \int dx_1 dx_2 exp[-i(p_1 x_1 - p_2 x_2)/\hbar] < x_1 | \hat{\rho} | x_2 > . \tag{6}$$

Similar to a two body problem in mechanics, we can define new variables as $R = \frac{x_1+x_2}{2}$ and $r = x_1 - x_2$, and a similar change of variables in the momentum representation, i.e., $P = \frac{p_1+p_2}{2}$ and $p = p_1 - p_2$. It is simple to show that

$$p_1 x_1 - p_2 x_2 = Pr + pR. \tag{7}$$

By substituting (7) into (6) and changing the variables, we could get

$$< P + \frac{p}{2}|\rho|P - \frac{p}{2} >= \frac{-1}{2\pi\hbar} \int\int drdRexp[-i(Pr + pR)/\hbar] < R + \frac{r}{2}|\rho|R - \frac{r}{2} > . \tag{8}$$

Doing a close look at the above relation it is just the Fourier transform of $\rho(R, r)$, where by analogy to the two particle problem, we can call R the center of mass coordinate, and r the relative coordinate. Because, we are interested to get a function containing both momentum and coordinate, we could either drop the Fourier transformation on relative coordinate to get the Wigner function, or drop the Fourier transformation on the center of mass coordinate to get the Shirley [4] function. Groot has presented another equivalent method for deriving the Wigner function [5]. By inserting one's we can get

$$A = \int dp'dp''dq'dq''|q'' >< q''|p'' >< p''|A|p' >< p'|q' >< q'|. \tag{9}$$

Then, we can introduce the new variables $p' = p - u/2$, $p'' = p + u/2$, $q' = q - v/2$ and $q'' = q + v/2$, where the Jacobian is equal to one, and use the relation $< q|p >= h^{-n/2} e^{\frac{i}{\hbar} p \cdot q}$ to get

$$\begin{aligned} A = h^{-n} \textstyle\int dpdqdudv|q + v/2 > e^{\frac{i}{\hbar}(p+u/2)(q+v/2)} < p + u/2|A|p - u/2 > \\ e^{\frac{-i}{\hbar}(p-u/2)(q-v/2)} < q - v/2|. \end{aligned} \tag{10}$$

This relation simplifies to

$$A = h^{-n} \int dpdqdudv|q+v/2 >< p+u/2|A|p-u/2 >< q-v/2|e^{\frac{i}{\hbar}(qu+pv)}. \tag{11}$$

By defining the $\hat{A}$ dependent function

$$a(p, q) = \int du < p + u/2|A|p - u/2 > e^{\frac{i}{\hbar} qu}, \tag{12}$$

and the $\hat{A}$ independent function

$$\Delta(p, q) = \int dv|q + v/2 >< q - v/2|e^{\frac{i}{\hbar} pv}. \tag{13}$$

We have

$$A = h^{-n} \int dpdqa(p, q)\Delta(p, q). \tag{14}$$

It is clear that $a(p, q)$ is the Wigner function corresponding to the operator $\hat{A}$. This is a natural way one can lead to the definition of the Wigner function.

4 Weyl operator

Years before this work by Wigner, Weyl [6] had proposed a method to construct an operator $\hat{A}$ corresponding to the phase space function $A(q,p)$. First we define

$$\alpha(\sigma,\tau) = \left(\frac{1}{2\pi\hbar}\right)^n \int dq \int dp e^{-i(\sigma q+\tau p)/\hbar} A(q,p) \tag{15}$$

and then,

$$\hat{A}(\hat{q},\hat{p}) = \int d\sigma \int d\tau \alpha(\sigma,\tau) e^{i(\sigma\hat{q}+\tau\hat{p})/\hbar} \tag{16}$$

Wigner's recipe is exactly the inverse of the Weyl's. If this is a suitable correspondence between $A(p,q)$ and $\hat{A}$, so we must be able to get the correct expectation value for $\hat{A}$ by use of $A(p,q)$, i.e.,

$$< \psi|\hat{A}|\psi >= \int dq \int dp P(q,p) A(q,p). \tag{17}$$

Before proving this equality, I should mention a lemma.

lemma 1: By using the Baker-Hausdorff lemma, we can prove that

$$e^{\hat{A}+\hat{B}} = e^{\hat{A}} e^{\hat{B}} e^{\frac{-1}{2}[A,B]}, \tag{18}$$

which yields to

$$e^{\frac{i}{\hbar}(\sigma\hat{q}+\tau\hat{p})} = e^{\frac{i}{\hbar}\sigma\hat{q}} e^{\frac{i}{\hbar}\tau\hat{p}} e^{i\sigma\tau/2}. \tag{19}$$

By substituting $A(p,q)$ from (15) and $\hat{A}$ from (16) into (17), we get

$$\begin{aligned} &\int d\sigma \int d\tau \alpha(\sigma,\tau) < \psi|e^{i(\sigma\hat{q}+\tau\hat{p})/\hbar}|\psi > \\ &= \int d\sigma \int d\tau \int dq \int dp P(q,p) e^{i(\sigma q+\tau p)/\hbar} \alpha(\sigma,\tau), \end{aligned} \tag{20}$$

which easily simplifies to

$$\begin{aligned} &< \psi|e^{i(\sigma\hat{q}+\tau\hat{p})/\hbar}|\psi >= \int dq \int dp P(q,p) e^{i(\sigma q+\tau p)/\hbar} \\ &= (2\pi\hbar)^{-n} \int dy \int dq \int dp \psi(q+y)^* \psi(q-y) e^{i(2py+\sigma q+\tau p)/\hbar}. \end{aligned} \tag{21}$$

The integral over p gives $(2\pi\hbar)^n \delta(2y+\tau)$, which allow us to perform the integral over y in order to get for the right hand side

$$\int dq \psi(q+\tau/2)^* \psi(q-\tau/2) e^{i(\sigma q)/\hbar}. \tag{22}$$

According to the lemma 1, the left hand side is

$$e^{i\sigma\tau/2} < \psi|e^{\frac{i}{\hbar}\sigma\hat{q}} e^{\frac{i}{\hbar}\tau\hat{p}}|\psi > . \tag{23}$$

Because p is the generator of translation (23) is equal to

$$\int dx e^{\frac{i}{\hbar}(\sigma x+\sigma\tau/2)} \psi(x)^* \psi(x+\tau). \tag{24}$$

By imposing the change of variable $x \to q-\tau/2$, we get the relation (22), so Q.E.D.

5 Properties of the Wigner distribution

A number of properties have been mentioned for this function [7]

(i) Since $P(q,p)$ should be real, it should be corresponding to a Hermitian operator, i.e.,

$$P(q,p) =< \psi|M(q,p)|\psi >, \tag{25}$$

where $M = M^\dagger$, i.e., Hermitian.

(ii)

$$\begin{aligned}\int dpP(q,p) = (\tfrac{1}{\pi\hbar})^n \int dp \int dy < q+y/2|\rho|q-y/2 > e^{ipy/\hbar} \\ = \int dy\delta(y) < q+y/2|\rho|q-y/2 >= |\psi(q)|^2 =< q|\rho|q > .\end{aligned} \tag{26}$$

$$\int dqP(q,p) = |\psi(p)|^2 =< p|\rho|p > \tag{27}$$

$$\int dq \int dpP(q,p) = Tr(\rho) = 1 \tag{28}$$

Derivation of the second and the third one are similar to that of the first one.

(iii) Translation of $P(q,p)$ in the momentum and coordinate spaces occur in accordance with the translation of the wave function, i.e., if $\psi(q) \to \psi(q+a)$ then $P(q,p). \to P(q+a,p)$, and if $\psi(q) \to e^{ip'q/\hbar}\psi(q)$ then $P(q,p) \to P(q,p-p')$ (iv) $P(q,p)$ should change the same way as ψ in space reflections and time inversions , i.e., if $\psi(q) \to \psi(-q)$, then $P(q,p) \to P(-q,-p)$ and, if $\psi(q) \to \psi(q)^*$ then $P(q,p) \to P(q,-p)$ (v) When the third and all higher order derivatives of the potential are zero we get the classical equations of motion (the Liouville equation). (This will be shown in the section *Dynamics*.)

(vi)

$$| < \psi(q)|\phi(q) > |^2 = 2\pi\hbar \int dq \int dpP_\psi(q,p)P_\phi(q,p) \tag{29}$$

(vii)

$$\int dq \int dpA(q,p)B(q,p) = 2\pi\hbar Tr(AB), \tag{30}$$

where $A(q,p)$ is the classical function corresponding to the quantum operator A. Using the property (ii) it can be easily shown that if $h(q,p) = f(q) + g(p)$ then we can get the expectation value of h by $\int\int dpdqP(q,p)[f+g]$

6 The Product of two Operators

Groenewold in a fundamental work argued foundational issues of quantum mechanics, which where discussed until 1946. He depicts the physical properties corresponding to the quantum mechanical operators $\hat{A}$ and $\hat{B}$ with a and b. He used the von Newman's assumptions, i.e., (I) if a corresponds to $\hat{A}$ and b corresponds to $\hat{B}$ then $a+b$ corresponds to $\hat{A}+\hat{B}$, and (II) if a corresponds to

$\hat{A}$ then $f(a)$ corresponds to $f(\hat{A})$. Then it is shown that such symbols constitute two isomorphic groups. Thus, if $\hat{A}$ and $\hat{B}$ do not commute then a and b should not commute. It can be shown that if we assume a and b as commuting observables we get into contradiction with assumptions (I) and (II). Therefore, a quantum system can not possess two physical properties corresponding to two non-commuting operators, and there is no reason to introduce different notation for operator and physical property. In that paper for the first time, he show that

$$\hat{A}\hat{B} = \hat{F} \to F(q,p) = A(q,p)e^{(\hbar\Lambda/2i)}B(q,p) = B(q,p)e^{-(\hbar\Lambda/2i)}A(q,p), \tag{31}$$

where

$$\Lambda = \frac{\overleftarrow{\partial}}{\partial p}\frac{\overrightarrow{\partial}}{\partial q} - \frac{\overleftarrow{\partial}}{\partial q}\frac{\overrightarrow{\partial}}{\partial p}. \tag{32}$$

Note that there is a dot product between the differentiation toward right and differentiation toward left. It is easy to see that $A(p,q)\Lambda B(p,q)$ is equivalent to $\{A(p,q), B(p,q)\}$, the Poisson bracket of A and B. By taking the matrix elements of (16), we get

$$< q''|\hat{A}|q' >= \int d\sigma \int d\tau \alpha(\sigma,\tau) < q''|e^{i(\sigma\hat{q}+\tau\hat{p})/\hbar}|q' > . \tag{33}$$

and by using the lemma 1, we can get

$$\begin{aligned} < q''|\hat{A}|q' > &= \int d\sigma \int d\tau \alpha(\sigma,\tau) e^{i\sigma\tau/2\hbar} e^{i\sigma(q'-\tau)/\hbar}\delta(q'-\tau-q'') \\ &= \int d\sigma \alpha(\sigma, q'-q'') e^{i\sigma(q'+q'')/2\hbar} d\sigma. \end{aligned} \tag{34}$$

Now we have

$$\begin{aligned} F(q,p) &= \int dz e^{ipz/\hbar} < q - \tfrac{z}{2}|\hat{A}\hat{B}|q+\tfrac{z}{2} > \\ &= \int dz \int dq' e^{ipz/\hbar} < q-\tfrac{z}{2}|\hat{A}|q' >< q'|\hat{B}|q+\tfrac{z}{2} > \\ &\int dz \int dq' \int d\sigma \int d\sigma' e^{(i/2\hbar)\sigma(q'+q-\frac{z}{2})} e^{(i/2\hbar)\sigma'(q'+q+\frac{z}{2})} \\ &\times\alpha(\sigma, q'-q+\tfrac{z}{2})\beta(\sigma', q-q'+\tfrac{z}{2})e^{ipz/\hbar}. \end{aligned} \tag{35}$$

By defining the new variables $\tau = q' - q + \frac{z}{2}$ and $\tau' = q - q' + \frac{z}{2}$, we would get

$$\begin{aligned} F(q,p) &= \int d\tau \int d\tau' \int d\sigma \int d\sigma' \alpha(\sigma,\tau) e^{(i/\hbar)(\sigma q+\tau p)} e^{(i/2\hbar)(\sigma'\tau-\sigma\tau')} \\ &\times e^{(i/\hbar)(\sigma' q+\tau' p)}\beta(\sigma',\tau'). \end{aligned} \tag{36}$$

Now consider the exponential between the other two exponentials and Taylor expand it. Consider the second term while forget about all constants, i.e., $\sigma'\tau - \sigma\tau'$ it is easy to see that by differentiation of the exponential on the right with respect to p and the exponential on the left with respect to q, we can get $\sigma'\tau$. We can get $\sigma\tau'$ by differentiation of the exponential on the right with respect to q and the exponential on the left with respect to p. Therefore, replacement of $(i/\hbar)(\sigma'\tau - \sigma\tau')$ by $(\hbar\Lambda/2i)$ makes no difference up to the second term in the Taylor expansion, by more elaboration you can show that this is also true for the higher order terms. After the mentioned replacement the definitions of $A(p, q)$ and $B(p, q)$ will appear in (36), and we will get the first equality in (31). If we change the place of the first two and the last two terms in (36), we can repeat the preceding discussion by interchanging the differentiation with respect to p by the differentiation with respect to q and vice versa. Thus, we can easily get the second equality in (31).

Now let me introduce another way of writing the product of two operators. Bopp operators are defined as [**?**]

$$Q = q - \frac{\hbar}{2i}\frac{\partial}{\partial p}, \qquad\qquad P = p + \frac{\hbar}{2i}\frac{\partial}{\partial q}. \tag{37}$$

By taking a test function f and a little elaboration you can show that $[\sigma q + \tau p, \tau\frac{\partial}{\partial q} - \sigma\frac{\partial}{\partial p}] = 0$. This equality yields to the following equality

$$exp\left\{\frac{i}{\hbar}[\sigma(q - \frac{\hbar}{2i}\frac{\partial}{\partial p}) + \tau(p + \frac{\hbar}{2i}\frac{\partial}{\partial q})]\right\} = e^{\frac{i}{\hbar}(\sigma q+\tau p)}e^{\frac{1}{2}(\tau\frac{\partial}{\partial q} - \sigma\frac{\partial}{\partial p})}. \tag{38}$$

If we multiply both sides by $e^{(\frac{i}{\hbar})(\sigma' q+\tau' p)}$, Taylor expand the middle term on the right hand side, and operate it on the exponential on its right, then every $\frac{\partial}{\partial p}$ will be replaced by τ' and every $\frac{\partial}{\partial q}$ will be replaced by σ'. Then, we will have the Taylor expansion of an exponential function in the middle, which can be gathered and give the final relation

$$\begin{gathered} exp\left\{\tfrac{i}{\hbar}[\sigma(q - \tfrac{\hbar}{2i}\tfrac{\partial}{\partial p}) + \tau(p + \tfrac{\hbar}{2i}\tfrac{\partial}{\partial q})]\right\} e^{(fraci\hbar(\sigma' q+\tau' p)} \\ = e^{\frac{i}{\hbar}(\sigma q+\tau p)}e^{\frac{i}{2\hbar}(\tau\sigma' - \sigma\tau')}e^{\frac{i}{\hbar}(\sigma' q+\tau' p)}. \end{gathered} \tag{39}$$

On the right hand side of (39), we have all the exponential terms we had on the right hand side of (36). By replacing them and using the notation introduced in (37), we get

$$F(q, p) = \int d\tau \int d\tau' \int d\sigma \int d\sigma' \alpha(\sigma, \tau)e^{\frac{i}{\hbar}(\sigma Q+\tau P)}e^{\frac{i}{\hbar}(\sigma' q+\tau' p)}\beta(\sigma', \tau'). \tag{40}$$

Now, we can define

$$\tilde{A}(Q, P) \equiv \int d\tau \int d\sigma\alpha(\sigma, \tau)e^{\frac{i}{\hbar}(\sigma Q+\tau P)}. \tag{41}$$

Therefore, we can express $F(p, q)$ as

$$F(p, q) = \tilde{A}(Q, P)B(p, q). \tag{42}$$

Similarly it can be shown that

$$F(p,q) = \tilde{B}(Q^*, P^*)A(p,q), \tag{43}$$

where

$$Q^* = q + \frac{\hbar}{2i}\frac{\partial}{\partial p}, \qquad P^* = p - \frac{\hbar}{2i}\frac{\partial}{\partial q}. \tag{44}$$

We know that the Wigner function is the function associated with $(1/2\pi\hbar)^n\hat{\rho})$, and we know that the equation of motion for ρ is

$$i\hbar\partial\hat{\rho}/\partial t = [\hat{H}, \hat{\rho}]. \tag{45}$$

Using the product rule just mentioned, we can transform (45) to

$$i\hbar\partial P/\partial t = H(q,p)e^{\hbar\Lambda/2i}P(q,p) - P(q,p)e^{\hbar\Lambda/2i}H(q,p). \tag{46}$$

The first term of the Taylor expansion is $HP - PH$, which is equal to zero. The second term in the Taylor expansion of the first and the second term of (46) are just negative of each other so they build up to $\frac{\hbar}{i}[\frac{\partial H}{\partial p}\frac{\partial P}{\partial q} - \frac{\partial H}{\partial q}\frac{\partial P}{\partial p}]$. For the third term we need

$$\Lambda^2 = (\frac{\overleftarrow{\partial}}{\partial p}\frac{\overrightarrow{\partial}}{\partial q} - \frac{\overleftarrow{\partial}}{\partial q}\frac{\overrightarrow{\partial}}{\partial p})(\frac{\overleftarrow{\partial}}{\partial p}\frac{\overrightarrow{\partial}}{\partial q} - \frac{\overleftarrow{\partial}}{\partial q}\frac{\overrightarrow{\partial}}{\partial p}) \tag{47}$$

By inserting two test functions f and g, respectively, in the left and right hand side of the expression in (47), we can show that

$$\Lambda^2 = \frac{\overleftarrow{\partial^2}}{\partial p^2}\frac{\overrightarrow{\partial^2}}{\partial q^2} - 2\frac{\overleftarrow{\partial^2}}{\partial q\partial p}\frac{\overrightarrow{\partial^2}}{\partial p\partial q} + \frac{\overleftarrow{\partial^2}}{\partial q^2}\frac{\overrightarrow{\partial^2}}{\partial p^2} \tag{48}$$

Because of the symmetry of (48), the third term in (45) is zero. Again, by inserting the test functions f and g, we can evaluate

$$\begin{aligned}\Lambda^3 = \Lambda^2\Lambda &= (\tfrac{\overleftarrow{\partial^2}}{\partial p^2}\tfrac{\overrightarrow{\partial^2}}{\partial q^2} - 2\tfrac{\overleftarrow{\partial^2}}{\partial q\partial p}\tfrac{\overrightarrow{\partial^2}}{\partial p\partial q} + \tfrac{\overleftarrow{\partial^2}}{\partial q^2}\tfrac{\overrightarrow{\partial^2}}{\partial p^2})(\tfrac{\overleftarrow{\partial}}{\partial p}\tfrac{\overrightarrow{\partial}}{\partial q} - \tfrac{\overleftarrow{\partial}}{\partial q}\tfrac{\overrightarrow{\partial}}{\partial p}) \\ &= \left(\tfrac{\overleftarrow{\partial^3}}{\partial p^3}\tfrac{\overrightarrow{\partial^3}}{\partial q^3} - 3\tfrac{\overleftarrow{\partial^3}}{\partial q\partial p^2}\tfrac{\overrightarrow{\partial^3}}{\partial p\partial q^2} + 3\tfrac{\overleftarrow{\partial^3}}{\partial q^2\partial p}\tfrac{\overrightarrow{\partial^3}}{\partial p^2\partial q} - \tfrac{\overleftarrow{\partial^3}}{\partial q^3}\tfrac{\overrightarrow{\partial^3}}{\partial p^3}\right),\end{aligned} \tag{49}$$

and

$$\begin{aligned}\Lambda^4 = \Lambda^3\Lambda &= \left(\tfrac{\overleftarrow{\partial^3}}{\partial p^3}\tfrac{\overrightarrow{\partial^3}}{\partial q^3} - 3\tfrac{\overleftarrow{\partial^3}}{\partial q\partial p^2}\tfrac{\overrightarrow{\partial^3}}{\partial p\partial q^2} + 3\tfrac{\overleftarrow{\partial^3}}{\partial q^2\partial p}\tfrac{\overrightarrow{\partial^3}}{\partial p^2\partial q} - \tfrac{\overleftarrow{\partial^3}}{\partial q^3}\tfrac{\overrightarrow{\partial^3}}{\partial p^3}\right)(\tfrac{\overleftarrow{\partial}}{\partial p}\tfrac{\overrightarrow{\partial}}{\partial q} - \tfrac{\overleftarrow{\partial}}{\partial q}\tfrac{\overrightarrow{\partial}}{\partial p}) \\ &= \left(\tfrac{\overleftarrow{\partial^4}}{\partial p^4}\tfrac{\overrightarrow{\partial^4}}{\partial q^4} - 4\tfrac{\overleftarrow{\partial^4}}{\partial q\partial p^3}\tfrac{\overrightarrow{\partial^4}}{\partial p\partial q^3} + 6\tfrac{\overleftarrow{\partial^4}}{\partial q^2\partial p^2}\tfrac{\overrightarrow{\partial^4}}{\partial p^2\partial q^2} - 4\tfrac{\overleftarrow{\partial^4}}{\partial q^3\partial p}\tfrac{\overrightarrow{\partial^4}}{\partial p^3\partial q} + \tfrac{\overleftarrow{\partial^4}}{\partial q^4}\tfrac{\overrightarrow{\partial^4}}{\partial p^4}\right).\end{aligned} \tag{50}$$

By continuing in this manner we can show that

$$\hbar\partial P/\partial t = -2H(q,p)sin(\hbar\Lambda/2)P(q,p). \tag{51}$$

In order to generalize the expression for Λ^2 to higher dimensions, we can write

$$\Lambda^2 = \left[\sum_i \left(\frac{\overleftarrow{\partial}}{\partial p_i}\frac{\overrightarrow{\partial}}{\partial q_i} - \frac{\overleftarrow{\partial}}{\partial q_i}\frac{\overrightarrow{\partial}}{\partial p_i}\right)\right]\left[\sum_j \left(\frac{\overleftarrow{\partial}}{\partial p_j}\frac{\overrightarrow{\partial}}{\partial q_j} - \frac{\overleftarrow{\partial}}{\partial q_j}\frac{\overrightarrow{\partial}}{\partial p_j}\right)\right], \tag{52}$$

in order to get

$$\Lambda^2 = \sum_{i,j}\left[\frac{\overleftarrow{\partial^2}}{\partial p_i \partial p_j}\frac{\overrightarrow{\partial^2}}{\partial q_i \partial q_j} - 2\frac{\overleftarrow{\partial^2}}{\partial q_i \partial p_j}\frac{\overrightarrow{\partial^2}}{\partial p_i \partial q_j} + \frac{\overleftarrow{\partial^2}}{\partial q_i \partial q_j}\frac{\overrightarrow{\partial^2}}{\partial p_i \partial p_j}\right]. \tag{53}$$

7 Proof of the impossibility of a positive phase space probability distribution

Since his first paper on this subject, Wigner was aware that this probability function gets negative values (unless the world was just made up of Gaussian wave packets). Thus, he emphasized that this is just a calculational tool, not a real probability distribution in phase space. Latter, Wigner published a paper in a book in the honor of Alfred Lande [8]; There he use the fact that for a mixed state we have $P(q,p) = \sum w_i P_i(q,p)$, where w_i is the probability of the i'th pure state, and $P_i(q,p)$ is the Wigner function for the i'th pure state. Then, he shows that by imposing the conditions (i) and (ii) (see section (5)) it is impossible to build an always positive distribution function. He used $\psi(q) = a\psi_1(q) + b\psi_2(q)$, where ψ_1 is zero outside I_1 and ψ_2 is zero outside of I_2. Consider I_1 and I_2 to be two non-overlapping intervals over the space of coordinate. Now we have

$$P_{ab}(q,p) = |a|^2 P_1 + a^* b P_{12} + ab^* P_{21} + |b|^2 P_2. \tag{54}$$

If q is outside of I_1, P_1 is zero for such a q and the only way to have a positive value for $P_{ab}(q,p)$ for every a and b is to have $P_{12}(q,p) = P_{21}(q,p) = 0$. The same reasoning can be given for the q outside of I_2. Therefore, every where, we have

$$P_{ab}(q,p) = |a|^2 P_1 + |b|^2 P_2. \tag{55}$$

This means that P_{ab} is independent of the complex phase of a/b, and it does not make sense. Consider the Fourier transform of ψ_1 and ψ_2 to be $\phi_1(p)$ and $\phi_2(p)$. By removing P_{ab} from equation (54) and equation (55), then integrating both sides of the resultant equation with respect to q and using the mentioned Fourier inverses, we get

$$\begin{gathered} |a|^2 \int P_1(q,p)dq + |b|^2 \int P_2(q,p)dq \\ = |a|^2|\phi_1(p)|^2 + |b|^2|\phi_2(p)|^2 + 2Re[ab^*\phi_1(p)\phi_2(p)^*]. \end{gathered} \tag{56}$$

For this relation to be valid for all a and b, we must have

$$\phi_1(p)\phi_2(p)^* = 0. \tag{57}$$

On the other hand, ϕ_1 and ϕ_2 are Fourier transforms of confined functions; thus, they cannot vanish on any finite interval. This is a contradiction, and QED. Because, it seems possible to break down every normalizable wave function into such a linear combination, thus this proof excludes the possibility of having a phase space distribution for a quantum state. Wigner [8], also, showed that by imposing the conditions (i)-(v) (1) is unique. While, O'connell and Wigner [9] show that by imposing conditions (i)-(iv) and (vi) ([**?**]) is the only possible distribution.

8 Dynamics of the Wigner function

If we want to express quantum mechanics in terms of the Wigner function we must derive Wigner functions equation of motion. This will be done with the aid of the Schroedinger equation, i.e.,

$$i\hbar\frac{\partial\psi(t)}{\partial t} = \left[-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial q^2} + V(q,t)\right]\psi(t). \tag{58}$$

By conjugate transposing both sides of the Schroedinger equation, we get

$$-i\hbar\frac{\partial\psi(t)^*}{\partial t} = \left[-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial q^2} + V(q,t)\right]\psi(t)^*. \tag{59}$$

Let me decompose the time dependence of P into two parts, i.e.,

$$\begin{aligned}\tfrac{\partial P}{\partial t} = (\tfrac{1}{\pi\hbar})^n \int dy \left[\tfrac{\partial\psi(q+y)^*}{\partial t}\psi(q-y) + \psi(q+y)^*\tfrac{\partial\psi(q-y)}{\partial t}\right]e^{2ipy/\hbar} \\ = \tfrac{\partial_k P}{\partial t} + \tfrac{\partial_v P}{\partial t}.\end{aligned} \tag{60}$$

In the last expression of (60), the first part arise from the kinetic part of the Hamiltonian and the second part arise from its potential part. By substituting (58) and (59) in (60) and considering the n to be equal to one, we can get

$$\frac{\partial_k P}{\partial t} = (\frac{-i}{2\pi m})\int dy\left[\frac{\partial^2\psi(q+y)^*}{\partial y^2}\psi(q-y) - \psi(q+y)^*\frac{\partial^2\psi(q-y)}{\partial y^2}\right]e^{2ipy/\hbar}, \tag{61}$$

where we have replaced $\partial^2/\partial q^2$ by $\partial^2/\partial y^2$. By partial integration, because ψ vanishes at $-\infty$ and ∞, we can get

$$\frac{\partial_k P}{\partial t} = (\frac{-p}{\pi\hbar m})\int dy\left[\frac{\partial\psi(q+y)^*}{\partial y}\psi(q-y) - \psi(q+y)^*\frac{\partial\psi(q-y)}{\partial y}\right]e^{2ipy/\hbar}. \tag{62}$$

By jumping back to $\partial/\partial q$, we obtain

$$\frac{\partial_k P}{\partial t} = (\frac{-p}{m})\frac{\partial P(q,p)}{\partial q}, \tag{63}$$

which is identical to the corresponding part of the classical Liouville equation. Also, we have

$$\begin{aligned}\frac{\partial_v P}{\partial t} &= \frac{i}{(\pi\hbar)^n\hbar}\int dy[(V\psi)(q+y)^*\psi(q-y) - \psi(q+y)^*(V\psi)(q-y)]e^{2ipy/\hbar} \\ &= \frac{i}{(\pi\hbar)^n\hbar}\int dy[V(q+y) - V(q-y)]\psi(q+y)^*\psi(q-y)e^{2ipy/\hbar}.\end{aligned} \quad (64)$$

By Taylor expanding V, we get

$$V(q+y) = \sum_{\lambda=0}^{\infty}\frac{y^\lambda}{\lambda!}\frac{\partial^\lambda V}{\partial q^\lambda}. \quad (65)$$

Therefore, we have

$$\frac{\partial_v P}{\partial t} = \frac{2i}{\pi\hbar^2}\int dy \sum_\lambda \frac{y^\lambda}{\lambda!}\frac{\partial^\lambda V}{\partial q^\lambda}\psi(q+y)^*\psi(q-y)e^{2ipy/\hbar}, \quad (66)$$

where the sum is over the odd positive integers λ, since the even terms resulting from $V(q+y)$ and those resulting from $V(q-y)$ cancel each other. Because by differentiating the exponential term with respect to p, we get a y multiplier, y^λ can be replaced with $[(\hbar/2i)(\partial/\partial p)]^\lambda$ to get

$$\frac{\partial_v P}{\partial t} = \sum_\lambda \frac{1}{\lambda!}(\frac{\hbar}{2i})^{\lambda-1}\frac{\partial^\lambda V(q)}{\partial q^\lambda}\frac{\partial^\lambda P(q,p)}{\partial p^\lambda}. \quad (67)$$

For the sake of simplicity, equations (65)-(67) are written for the one dimensional case ($n=1$), . In order to generalize them to higher dimensions $\lambda!$ should be replaced by $\Pi_i\lambda_i!$, any thing to power λ with the same thing to the power $\sum_i \lambda_i$, ∂q^λ with $\Pi_i\partial q_i^\lambda$, and ∂p^λ with $\Pi_i \partial p_i^{\lambda_i}$. Which is a trivial generalization; But, remember λ_i's take positive integers that yield an odd positive integer for $\sum_i \lambda_i$. We can, also, write $\frac{\partial_v P}{\partial t}$ in the form

$$\frac{\partial_v P}{\partial t} = \int dj P(q,p+j)J(q,j), \quad (68)$$

while

$$\begin{aligned}J(q,j) &= \frac{1}{(\pi\hbar)^n\hbar}\int dy[V(q+y) - V(q-y)]e^{-2ijy/\hbar} \\ &= \frac{i}{(\pi\hbar)^n\hbar}\int dy[V(q+y) - V(q-y)]sin(2jy/\hbar)\end{aligned} \quad (69)$$

has interpreted as the probability of a jump in momentum by an amount j if the position is q. We can go from the first to the second equality in (69), because $e^{ix} = cos(x) + isin(x)$ and the function in the square brackets is an odd function so when it is multiplied by an even function, $cos(2jy/\hbar)$ and integrated over the whole space, it will give zero. Now, we are able to get the equation of motion as

$$\frac{\partial P}{\partial t} = -\sum_{i=1}^{n}\frac{p_i}{m_i}\frac{\partial P}{\partial q_i} + \sum \frac{\partial^{\sum\lambda_i}V}{\Pi_i\partial q_i^{\lambda_i}}\frac{(\hbar/2i)^{\sum\lambda_i - 1}}{\Pi\lambda_i!}\frac{\partial^{\sum\lambda_i}P}{\Pi\partial p_i^{\lambda_i}}. \quad (70)$$

Now consider the case where the potential has no third or higher order derivative; then, evidently (69) is the classical Liouville equation. For a system consisting of a bunch of harmonic oscillators and free particles, we can solve the easier classical equations of motion and get the exact quantum result! This is surprising.

9 Attempts for giving a probability distribution interpretation to the Wigner function

Some people have argued this function as a valid probability distribution, and some others argue it as a valid probability distribution just for some situations. Stenholm has argued that, we can "obtain verifiable predictions" only by using "suitable test bodies." He show that always positive probabilities come out of the Wigner distribution when these arguments are implemented. In relativistic quantum mechanics [10], even if we are working in the position representation, the choice of the position observable is not at all trivial. Therefore, probably in non-relativistic cases it is just the absence of mathematical complications which make us to believe that we can make a classical interpretation of position and momentum.

The way to measure the momentum and coordinate of a particle is to let it interact with another body which we usually let approach it's classical limit. This second body is a test particle, and we are actually performing a scattering experiment. In this scattering experiment, test particle transfers the desired information out of the interaction region. "Only probability distributions observable in this manner can be given a physical interpretation" [3]. In cases where the test particle is carrying both coordinate and momenta information, restrictions due to Heisenberg uncertainty principle should be taken into account. Stenholm emphasized that, in order to get the most precise results for both momentum and coordinate, we must use a test particle which is in a state of minimum uncertainty. At the end, the test body is bringing out some information which at best allow us to confine our system to a region of phase space satisfying the relation $\Delta p \Delta q \leq \frac{\hbar}{2}$ and no more precise detail is achievable. The minimum uncertainty wave packet can be determined uniquely as

$$\psi_0(x) = Cexp\left(-\frac{(x-<x>)^2}{4b^2} + \frac{i<P>x}{\hbar}\right), \tag{71}$$

where $<>$ denotes the expectation value, and we have uncertainties $\Delta x = b$ and $\Delta p = \hbar/2b$. The Wigner function for this minimum uncertainty wave packet is

$$W_0(R,P) = Aexp\left(-\frac{(R-<x>)^2}{2b^2} - \frac{4b^2(P-<P>)^2}{2\hbar^2}\right). \tag{72}$$

Stenholm argued that "a Wigner function $W(P,R)$ is not directly observable but has to be convoluted with the function describing the test particle, which smears it, at least, by the amount implied by the function" (72). This convolution leads to

$$P(\pi,q) = A\int\int exp\left(-\frac{(R-q)^2}{2b^2} - \frac{2b^2(P-\pi)^2}{\hbar^2}\right)W(R,P)dRdP$$

$$= (\frac{A}{\pi\hbar})\int\int\int drdPdRe^{(-iPr/\hbar)}e^{-(R-q)^2/2b^2}e^{-2b^2(P-\pi)^2/\hbar^2}\psi(R+\frac{r}{2})\psi(R-\frac{r}{2})^*. \tag{73}$$

Here (q,π) are a couple of position and momentum coordinate, and Stenholm hope them to give meaningful phase space interpretation. A is just a normalization constant. By carrying out the integral over P, we can show that $P(q,\pi)\geq 0$. The immediate criticism to this approach is that not all test bodies are minimum uncertainty wave packets. Thus, Stenholm emphasize that in order for the test body to exhibit a nearly classical behavior it must be smooth and every wave function smoother than the minimum uncertainty wave packet will fulfill the positiveness requirement. All in all, we have to calculate $W(P,R;t)$ up to the moment of measurement, and then smooth it to obtain $P(\pi,q;t)$. Since, $P(\pi,q;t)$ depends on the test particle prepared by the observer it has no dynamics. The idea of smoothing the Wigner function with a Gaussian function was first introduced by Husimi [11]. He get the positive distribution which is now called the Husimi distribution. Husimi didn't interpret it as a phase space distribution because it doesn't poses the property (ii). There is also a bunch of work for interpreting Quantum Optics based on Wigner function. Including Marshall and Santos [12], who argue that, there is just a subset of states in the Hilbert space which can be generated in the laboratory, and those states "May be represented by a positive Wigner distribution." They have also claimed that [13], the experiments which are exhibiting non-classical behavior of light can be interpreted just by assuming light as an electromagnetic wave in accordance with the Maxwell's equations of motion. Also, Holland *et al* [14] published on the "Relativistic generalization of the Wigner function and its interpretation in the causal stochastic formulation of quantum mechanics." There are some good reviews on the mathematical properties and applications of the Wigner function, e.g., [7], [15] and [16].

10 Discussion

We can argue that, any experiment which is designed to measure both momentum and coordinate will get average information about a region of phase space which is large enough to give a positive value. The Wigner function give a positive average over this region and thus interpreting the Wigner function as a probability distribution function is experimentally adequate. But does it make sense to consider a probability function which has no meaning on a point but is representing a physical reality when averaged over any large enough interval! Thats weird but is it more weird than a particle owning either momentum or position and not both of them at the same time? The classical formalism of quantum mechanics doesn't provide any prediction regarding a simultaneous measurement of position and momentum. If it was possible to perform such an experiment, for example, as suggested, thorough sending out a test particle to interact with system (quantum particle) and be able to measure its position and momentum before and after the interaction, because it is approaching classical behavior, then we could perform an experimental test for this interpretation. But it seems that no semi-classical test particle can be used for a measurement on a quantum system without enormously changing its state.

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