1. ASTRONOMY

Magnitude System: The flux f_F (erg cm^{-2} s^{-1}) passing through filter F with a response $A_F(\lambda)$ (fraction transmitted, i.e., between 0 to 1 at wavelength λ) is related to the absolute flux f through

$$f_F = \int f_\lambda A_\lambda(\lambda) d\lambda, \quad f = \int f_\lambda d\lambda \tag{1}$$

Standard filters centred around $\lambda_U \approx 3650$Å, $\lambda_B \approx 4400$Å, $\lambda_V \approx 5480$Å. Definition of apparent/bolometric magnitude:

$$m_{F,1} - m_{F,2} = -2.5 \log_{10} \left(\frac{f_{F,1}}{f_{F,2}} \right) \tag{2}$$

$$m_1 - m_2 = -2.5 \log_{10} \left(\frac{f_1}{f_2} \right) \tag{3}$$

Normalization: $m = 0$ and $m_F = 0$ for the star Vega. Also, $m_\odot = -26.83$, $m_{Sirius} = -1.6$ and sensitivity of the eye is $m \lesssim 6$ (for pupil diameter of 8mm). With normalization, the magnitudes through standard filters (or the bolometric normalization) are:

$$m_X = -2.5 \log_{10} \left[\frac{\int f_\lambda(\lambda) A_X(\lambda) d\lambda}{F_{X,\lambda 0} \int A_X} \right] \tag{4}$$

$$m_{bol} = -2.5 \log_{10} \left[\frac{\int f_\lambda(\lambda) d\lambda}{F_0} \right] \tag{5}$$

where the normalizations fluxes are

$$F_0 = 2.52 \times 10^{-5} \text{ erg cm}^{-2} \text{ s}^{-1} \tag{6}$$

$$F_{X,\lambda 0} = F_{\{U,B,V,R,orI\},\lambda 0} = \tag{7}$$

$$\{4.27, 6.61, 3.64, 1.74, 0.832\} \times 10^{-9} \text{erg cm}^{-2}\text{s}^{-1}\text{Å}^{-1}$$

e.g., a flux of 6.61×10^{-9}erg cm^{-2}s^{-1}Å^{-1} at 4400Å gives $m_B = 0$, or a spectrally integrated flux of 2.52×10^{-5} erg cm^{-2} s^{-1} gives $m_{bol} = 0$.

Absolute (M) vs. Apparent magnitudes (m). $M = m$ for a $\star$ at $d = 10pc$, i.e.,

$$M = m - 5 \log_{10} \left(\frac{d}{10pc} \right). \tag{8}$$

Effective temperature is the black body temperature that gives L.

Color is the temperature magnitude difference in two filters. e.g.,

$$B - V \equiv m_B - m_V = \text{color index} \tag{9}$$

Color temperature is the blackbody temperature that gives the observed $B - V$.

Wien Approximation for blackbody $F \propto \exp(hc/\lambda kT)$ (at short wavelengths), giving:

$$B - V \approx \frac{7090K}{T_c} - 0.71 \tag{10}$$

For Vega, $B - V \equiv 0$ and $T_c \approx 10000$K.

Spectral Types: From hot ($\sim$ 50000K) to cold ($\sim$ 3000K): OBAFGKM. Each type goes from 0 to 9 (e.g., A0..A9). The Sun is a G2 star. Vega is A0.

Diffraction limit of a telescope with a diameter d at a frequency λ is:

$$\theta \text{min} \approx 1.2 \frac{\lambda}{d} \tag{11}$$

2. POTENTIAL ENERGY AND VIRIAL THEOREM

The gravitational potential energy of spherically symmetric mass M is:

$$U_{grab} = -G \int_0^M \frac{mdm}{r} \tag{12}$$

The virial theorem:

$$\frac{1}{2} \frac{d^2 I}{dt^2} = 2K + \sum_i \mathbf{F} \cdot \mathbf{r} \tag{13}$$

where I is the "spherical" moment of inertial $I \equiv \sum_i m_i r_i^2$. K is the kinetic energy of the system $K \equiv \sum_i m_i (d\mathbf{r}_i/dt) \cdot \mathbf{r}_i$. Particles in a gravitational field:

$$\mathbf{F}_{i,j} = -\frac{G m_i m_j}{r_{i,j}^3} (\mathbf{r}_i - \mathbf{r}_j) \tag{14}$$

And the virial theorem becomes:

$$K = -\frac{1}{2} \sum_{pairs} \mathbf{F}_{i,j} \cdot (\mathbf{r}_i - \mathbf{r}_j) = \frac{1}{2} \sum_{pairs} \frac{G m_i m_j}{r_{i,j}}$$

$$= \frac{1}{2} \int_0^M \frac{G m(r) dm}{r} = -\frac{\Omega}{2}. \tag{15}$$

3. EQUATION OF STATE OF IDEAL GAS

The equation of state of an ideal gas is

$$P = \frac{\rho k T}{\mu m_p} + \frac{1}{3} a T^4. \tag{16}$$

Here a is the radiation energy constant. It is related to the Stefan-Bolzmann constant through:

$$a = \frac{4\sigma}{c} \tag{17}$$

Note that if the matter is degenerate, then there will be another term. This pressure can be written with the help of the gas to total pressure ratio

$$\beta \equiv \frac{p_{gas}}{p_{tot}} \;\; ; \;\; P = \left[\left(\frac{k}{\mu m_p} \right)^4 \frac{3}{a} \frac{(1-\beta)}{\beta^4} \right]^{\frac{1}{3}} \rho^{4/3} \tag{18}$$

Molecular Weight μ appearing above is the average weight of a particle (contributing to the pressure) in units of the proton mass m_p. Given mass fractions n_j for specie j in the ionised plasma, the molecular weight is:

$$\mu = \frac{\sum_j n_j A_j}{\sum_j n_j (1 + Z_j)}. \tag{19}$$

Each specie has an atomic mass A_j and charge $Z_j e$. If one gram of matter contains X grams of hydrogen, Y grams of He and Z grams of the rest ("Metalicity"), one has:

$$\frac{1}{\mu} \approx 2X + \frac{3}{4} Y + \underbrace{\left\langle \frac{(1+Z_j)}{A_j} \right\rangle_{j \neq 1,2}}_{\approx 1/2} Z \tag{20}$$

4. DEGENERATE MATTER AND WHITE DWARFS

Non-relativistic. The degeneracy pressure is:

$$P_{e,nr} = \left[\frac{1}{20} \left(\frac{3}{\pi} \right)^{2/3} \frac{h^2}{m_e m_p^{5/3} \mu_e^{5/3}} \right] \rho^{5/3} \tag{21}$$

here $\mu_e = \rho/(n_e m_p)$ is the number of baryons per electron. Using $n = 1.5$ polytropic solution:

$$R = (1.22 \times 10^4 \mathrm{km}) \left(\frac{\rho_c}{10^6 \mathrm{g\ cm^{-3}}}\right)^{-1/6} \left(\frac{\mu_e}{2}\right)^{-5/6} \quad (22)$$

$$M = (0.4964 M_\odot) \left(\frac{\rho_c}{10^6 \mathrm{g\ cm^{-3}}}\right)^{1/2} \left(\frac{\mu_e}{2}\right)^{-5/2} \quad (23)$$

or

$$M = 0.7011 M_\odot \left(\frac{R}{10^4 \mathrm{km}}\right)^{-3} \left(\frac{\mu_e}{2}\right)^{-5} \quad (24)$$

Relativistic. The degeneracy pressure is:

$$P_{e,nr} = \left[\frac{1}{8}\left(\frac{3}{\pi}\right)^{1/3} \frac{hc}{m_p^{4/3}\mu_e^{4/3}}\right] \rho^{4/3} \quad (25)$$

Using $n = 3$ polytropic solution:

$$R = (3.347 \times 10^4 \mathrm{km}) \left(\frac{\rho_c}{10^6 \mathrm{g\ cm^{-3}}}\right)^{-1/3} \left(\frac{\mu_e}{2}\right)^{-2/3} \quad (26)$$

$$M = (1.457 M_\odot) \left(\frac{\mu_e}{2}\right)^{-2} \equiv M_{Ch} = 3.10 \left(\frac{\hbar c}{G}\right)^{3/2} \frac{1}{m_p^2 \mu_e^2} \quad (27)$$

5. EQUATIONS OF STELLAR STRUCTURE

Mass continuity

$$\frac{dm}{dr} = 4\pi r^2 \rho \quad (28)$$

Hydrostatic equation

$$\frac{dp}{dr} = -\frac{Gm}{r^2}\rho \quad (29)$$

Radiative Transfer

$$\left.\frac{dT}{dr}\right|_{rad} = -\frac{3}{4ac}\frac{\kappa_m \rho}{T^3}\frac{L}{4\pi r^2} \quad (30)$$

Energy transfer

$$\frac{dT}{dr} = \max\left(\left.\frac{dT}{dr}\right|_{rad}, \left.\frac{dT}{dr}\right|_{ad}\right) \quad (31)$$

(i.e., the least negative of the two gradients!) Nuclear Energy generation

$$\frac{dL}{dr} = 4\pi r^2 \rho \epsilon \quad (32)$$

here ϵ is the energy generation per unit mass.
Often, the opacity is approximated as:

$$\kappa_m = \tilde{\kappa} \rho^a T^b \quad (33)$$

For Thomson opacity, $a = b = 0$. For Kramer's opacity $a = 1, b = -3.5$.
While the energy generation is approximated as:

$$\epsilon = \tilde{\epsilon} \rho^m T^n \quad (34)$$

For all reactions $m = 1$ except 3α burning for which $m = 2$. $n \approx 4$ for pp, $n \approx 16$ for CNO burning, while $n \approx 40$ for 3α.
Eddington Luminosity: The ratio between the radiative transfer equation (written with p_{rad}) and the hydrostatic equation (for p_{tot}) gives:

$$1 - \beta \approx \frac{dp_{rad}}{dp_{tot}} = \frac{L}{L_{edd}} \quad (35)$$

where $\beta \equiv p_{gas}/p_{tot}$, and

$$L_{edd} \equiv \frac{4\pi G M c}{\kappa_m}. \quad (36)$$

Equations in terms of m for homology: Assuming radiative transfer, the stellar structure equations can be written with m as the coordinate:

$$\frac{dr}{dm} = \frac{1}{4\pi r^2 \rho} \quad (37)$$

$$\frac{dp}{dm} = -\frac{Gm}{4\pi r^4} \quad (38)$$

$$\frac{dT}{dm} = -\frac{3}{64\pi^2 ac}\frac{\kappa}{T^3}\frac{L}{r^4} \quad (39)$$

$$\frac{dL}{dm} = \epsilon \quad (40)$$

6. CONVECTION

Necessary condition for convection

$$\left.\frac{dT}{dr}\right|_{atmos} < \left.\frac{dT}{dr}\right|_{adiab} = -\frac{g}{c_p} = -\frac{g\mu m_p}{k}\left(\frac{\gamma - 1}{\gamma}\right) \quad (41)$$

(i.e., atmospheric gradient is steeper). In stars, it becomes:

$$\frac{3}{16\pi}\frac{\kappa_m \rho L(r)}{acT^3} > \frac{Gm(r)\mu m_p}{k}\left(\frac{\gamma - 1}{\gamma}\right) \quad (42)$$

7. RADIATIVE TRANSFER

Mean free path of photons ℓ is related to scattering (or absorption cross-section) and number density of scatterers n, to extinction (opacity per unit volume), or to opacity per unit mass through:

$$\ell^{-1} = n\sigma = \kappa_v = \rho \kappa_m \quad (43)$$

Number of diffusion steps needed to cross a length d is $\sim (d/l)^2$, over a time scale $\sim d^2/(lc)$.
Intensity I is defined as the flux (energy per unit area per unit time) per unit solid angle. The flux along a ray s satisfies (under the gray approximation)

$$\frac{dI}{ds} = -\kappa_v \left(I - \frac{\sigma T^4}{\pi}\right). \quad (44)$$

If there is only absorption (and re-emission in another waveband), then the intensity decays exponentially:

$$I = I_0 \exp(-\kappa_v s). \quad (45)$$

Thus, the probability that photons arrive from a unit interval at a given distance s is

$$P(s)ds = \kappa_v \exp(-\kappa_v s) ds. \quad (46)$$

Optical depth between r_1 and r_2 is defined through

$$\tau \equiv \int_{r_1}^{r_2} \kappa_m \rho dr \quad (47)$$

Often one takes $r_2 = \infty$ to get the optical depth to infinity.
The equation of radiative transfer (which can be obtained by integrating over I formed from a background with a gradient temperature).

$$F = -\frac{ac}{3}\frac{\nabla(T^4)}{\kappa_v} = -\frac{c}{3}\frac{\nabla E}{\kappa_v}. \quad (48)$$

Blackbody radiation field. Planck's law describes the black body radiation (energy per unit area per solid angle) from a surface at temperature T, either per unit frequency:

$$B_\nu(T) = \frac{2h\nu^3}{c^2} \frac{1}{\exp(h\nu/kT) - 1} \tag{49}$$

or per unit wavelength:

$$B_\lambda(T) = \frac{2hc^2}{\lambda^5} \frac{1}{\exp(hc/\lambda kT) - 1} \tag{50}$$

8. NUCLEAR REACTIONS

The rate of the reaction $a + X \to b + Y + Q$ can be written as:

$$r_{aX} = \frac{1}{(1+\delta_{aX})} \frac{\rho^2 N_A^2 X_a X_X}{A_a A_X} \langle \sigma v \rangle \tag{51}$$

with X_i and A_i the mass fractions and atomic weight of specie i, and

$$\langle \sigma v \rangle = \frac{(8/\mu\pi)^{\frac{1}{2}}}{(kT)^{3/2}} \int_0^\infty S(E) \exp(-\frac{E}{kT} - \frac{b}{\sqrt{E}}) dE \tag{52}$$

$S(E)$ is generally a slowly varying function of E which depends on the reaction. And

$$b \equiv \frac{\sqrt{2\mu}\pi Z_a Z_X e^2}{\hbar} \tag{53}$$

Here μ is the reduced mass. The Gamov Peak is:

$$E_0 = \left(\frac{bkT}{2}\right)^{2/3} = 1.2 \left(Z_a^2 Z_X^2 A_{red} T_6^2\right)^{1/3} keV \tag{54}$$

For $S(E) = S_0 = const$, one obtains:

$$r_{ax} = \frac{n_a n_x}{A_{red} Z_a Z_x} 7 \times 10^{-19} \frac{S_0}{[keV \cdot \text{barn}]} \tau^2 e^{-\tau} s^{-1} cm^{-3} \tag{55}$$

with A_{red} being the reduced atomic weight and

$$\tau \equiv \frac{3E_0}{kT} = 42.5 \left(\frac{Z_a^2 Z_x^2 A_{red}}{T/10^6 K}\right)^{1/3} \tag{56}$$

9. POLYTROPIC STARS

Polytropic approximation:

$$P = K\rho^\Gamma \equiv K\rho^{(n+1)/n}. \tag{57}$$

In adiabatic gas: $\Gamma = \gamma = c_p/c_v$. In non-relativistic monoatomic gas $\gamma = 5/3$, $n = 1.5$. In relativistic monoatomic gas: $\gamma = 4/3$, $n = 3$. Eddington standard model $n = 3$.
Standard transformation to get Lane-Emden equation:

$$\rho = \rho_c \phi^n \;\; ; r = \xi \ell \tag{58}$$

$$\ell \equiv \left[\frac{(n+1)K\rho_c^{(1-n)/n}}{4\pi G}\right]^{1/2} \tag{59}$$

which is:

$$\frac{1}{\xi^2} \frac{d}{d\xi} \left(\xi^2 \frac{d\phi}{d\xi}\right) = -\phi^n \tag{60}$$

TABLE 1
VALUES OF ξ_1 AND $\Upsilon(2, n)$, FOR VARIOUS POLYTROPIC INDICES n.

n	ξ_1	$\Upsilon(2,n)$
0	2.44949	4.8988
1	3.14159	3.14159
1.5	3.65375	2.71406
2	4.35287	2.41105
3	6.89685	2.01824
4	14.9715	1.79723

TABLE 2
VALUES OF $\int_0^{\xi_1} \xi^i \phi^j(\xi) d\xi$ FOR $n = 3$ POLYTROPE.

	i=0	i=1	i=2	i=3	i=4	i=5	i=6
j=0	6.896	23.783	109.353	565.643	3120.93	17937.1	106037.
j=1	2.662	4.847	14.191	52.465	222.832	1036.23	5132.92
j=2	1.757	2.132	4.327	11.667	37.943	140.807	575.485
j=3	1.38	1.292	2.018	4.223	10.851	32.515	109.748
j=4	1.168	0.914	1.181	2.036	4.317	10.748	30.454
j=5	1.03	0.704	0.788	1.169	2.126	4.537	11.05
j=6	0.931	0.571	0.57	0.751	1.207	2.268	4.858
j=7	0.856	0.48	0.437	0.521	0.755	1.277	2.454
j=8	0.796	0.414	0.348	0.382	0.507	0.784	1.374
j=9	0.748	0.364	0.285	0.291	0.359	0.514	0.832
j=10	0.707	0.325	0.239	0.23	0.265	0.354	0.535
j=15	0.571	0.211	0.124	0.094	0.085	0.088	0.103
j=20	0.492	0.156	0.078	0.05	0.039	0.034	0.033
j=25	0.439	0.123	0.055	0.031	0.021	0.016	0.014

with boundary conditions: $\phi(0) = 1$ and $d\phi/d\xi_0 = 0$. Outer boundary exists only for $n < 5$, which is $\phi(\xi_1) = 0$. Stellar radius:

$$R_\star = \xi_1 \ell = \left[\frac{(n+1)K}{4\pi G}\right]^{\frac{1}{2}} \rho_c^{(1-n)/2n} \xi_1 \tag{61}$$

Many interesting variables can be written with dimensionless integrals of the form:

$$\Upsilon(i,j) = \int_0^{\xi_1} \xi^i \phi^j(\xi) d\xi \tag{62}$$

For example, the stellar mass:

$$M_\star = 4\pi\ell^3 \rho_c \Upsilon(2,n) = -4\pi\ell^3 \rho_c \, \xi_1^2 \left.\frac{d\phi}{d\xi}\right|_{\xi_1} \tag{63}$$

Average density

$$\frac{\bar{\rho}}{\rho_c} = -\frac{3}{\xi_1} \left.\frac{d\phi}{d\xi}\right|_{\xi_1} \tag{64}$$

10. ACCRETION DISKS

The rate of energy release in accretion disks from infinity down to a radius r is roughly

$$L(r) \approx \frac{GM\dot{m}}{2r} \tag{65}$$

with $\dot{m}$ being the mass accretion rate. This assumes a Keplerian disk such at the velocity is nearly Keplerian at each radius. The energy dissipated at each radius (per unit area) is therefore:

$$D(r) = \frac{1}{(4\pi r)} \frac{dL(r)}{dr} \approx \frac{GM\dot{m}}{8\pi r^3} \tag{66}$$

11. SUPERNOVA EXPLOSIONS

If we look at a spherical shell having a radial width $\Delta r = fr$, with $r = vt$, then the diffusion time scale is

$$t_{\rm diff} \sim \frac{f^2 r^2}{\ell c} \sim \frac{f \kappa_m m_{\rm env}}{4\pi c v t} \tag{67}$$

where κ_m is the opacity per unit mass and $m_{\rm env}$ is the mass of the shell.

TABLE 3
USEFUL CONSTANTS

Speed of light	c	2.998 10^{10}	cm s^{-1}
Planck's constant	h	6.626 10^{-27}	erg s
	$\hbar$	1.055 10^{-27}	erg s
Bolzmann's const	k	1.381 10^{-16}	erg K^{-1}
Electron charge	e	4.803 10^{-10}	esu
Electron rest mass	m_e	9.110 10^{-28}	g
Proton rest mass	m_p	1.673 10^{-24}	g
Gravitational const	G	6.673 10^{-8}	dyne cm^2 g^{-2}
Avogadro's number	N_A	6.022 10^{23}	$mole^{-1}$
Gas constant	R	8.314 10^{7}	erg K^{-1} $mole^{-1}$
Stephan-Boltzmann	σ	5.670 10^{-5}	erg cm^{-2} s^{-1} K^{-4}
Radiation constant	a	7.564 10^{-15}	erg cm^{-3} K^{-4}
Thomson x-section	σ_T	6.656 10^{-25}	cm^2
Electron-Volt	eV	1.602 10^{-12}	erg
Astronomical unit	AU	1.496 10^{13}	cm
Parsec	pc	3.086 10^{18}	cm
Light year	l.y.	9.460 10^{17}	cm
Solar mass	$M_\odot$	1.989 10^{33}	g
Solar radius	$R_\odot$	6.960 10^{10}	cm
Solar luminosity	$L_\odot$	3.826 10^{33}	erg s^{-1}
Solar magnitude:			
apparent bolometric	$m_{b,\odot}$	-26.75	
absolute bolometric	$M_{b,\odot}$	+4.82	
Solar Metalicity	Z	0.02	
Lunar apparent mag.	$m_{V,☾}$	-12.74	

12. COSMOLOGY

Robertson-Walker Metric

$$ds^2 = (cdt)^2 - a(t)^2 \left[\frac{dr^2}{1 - Kr^2} + r^2 (d\theta^2 + \sin^2\theta d\phi^2) \right] \tag{68}$$

K denotes the curvature. $a(t)$ is the scale factor[1]. $K = -1, 0, +1$ for hyperbolic, flat and elliptical metrics.
Null Geodesics ($ds = 0$, photons) satisfy

$$\int_{t_1}^{t_0} \frac{cdt}{a(t)} = \int_0^{r_1} \frac{dr}{(1 - Kr^2)^{1/2}} \tag{69}$$

$$\equiv f(r_1) = \begin{cases} \sinh^{-1}(r) & K = -1 \\ r & K = 0 \\ \sin^{-1}(r) & K = +1 \end{cases}$$

Redshift:

$$1 + z = \frac{\lambda_{\rm obs}}{\lambda_{\rm emitted}} = \frac{a_{\rm obs}}{a_{\rm emitted}} \tag{70}$$

(obs = present day, emitted = some time in the past).

2nd order approximation:

$$a(t) = a_0 \left[1 + H_0 (t - t_0) - \frac{1}{2} q_0 H_0^2 (t - t_0)^2 + \mathcal{O}(t - t_0)^3 \right] \tag{71}$$

[1] For example, for a closed universe it would be the radius of the 4-sphere, the surface of which is the 3-volume of the universe.

$$q_0 = -\frac{\ddot{a}(t_0) a(t_0)}{\dot{a}(t_0)^2} \tag{72}$$

Taking $a_0/a(t)$ gives

$$z = H_0 (t_0 - t) + \left(1 + \frac{q_0}{2}\right) H_0^2 (t_0 - t)^2 + ... \tag{73}$$

or inverted, the look back time is:

$$(t_0 - t) = \frac{1}{H_0} \left(z - \left(1 + \frac{q_0}{2}\right) z^2 + \mathcal{O}(z^3) \right) \tag{74}$$

If we look at a light ray ($ds = 0$) then $\int_t^{t_0} (c/a(t)) dt$ gives

$$r = \frac{c}{a_0} \left[(t_0 - t) + \frac{1}{2} H_0 (t_0 - t)^2 + ... \right] \tag{75}$$

$$= \frac{c}{a_0 H_0} \left[z - \frac{1}{2} (1 + q_0) z^2 + ... \right] \tag{76}$$

Luminosity Distance: Defined as

$$d_L \equiv \sqrt{\frac{L}{4\pi F}} \tag{77}$$

To second order in z:

$$d_L \equiv \frac{c}{H_0} \left[z + \frac{1}{2} (1 - q_0) z^2 + \mathcal{O}(z^3) \right] \tag{78}$$

Friedman equations:

$$\ddot{a} = -\frac{4\pi G}{3} \left(\rho + 3\frac{p}{c^2} \right) a + \frac{\Lambda c^2 a}{3} \tag{79}$$

$$\dot{a}^2 = \frac{8\pi G}{3} \rho a^2 - K c^2 + \frac{\Lambda c^2}{3} a^2 \tag{80}$$

Cosmological parameters:

$$H_0 = \left(\frac{\dot{a}}{a} \right)_0 \tag{81}$$

$$\Omega_0 = \left(\frac{\rho}{\rho_c} \right)_0 \tag{82}$$

$$\rho_c = \frac{3 H_0^2}{8\pi G} \tag{83}$$

$$q_0 = -\left(\frac{\ddot{a} a}{\dot{a}^2} \right)_0 \tag{84}$$

Cosmological equation of state

$$p = w \rho c^2 \tag{85}$$

For pressure less "dust", $w = 0$ (i.e., non-relativistic gas). For relativistic gas (e.g., radiation) $w = 1/3$.
From first law of thermodynamics:

$$d(\rho c^2 a^3) = -p dV = -3 w \rho a^2 da \tag{86}$$

Integration gives

$$\rho a^{3(1+w)} = const \tag{87}$$

Thus, $\rho_m = \rho_{m,0} (1+z)^3$ and $\rho_{rad} = \rho_{rad,0} (1+z)^4$