Linear Perturbation Theory: Non-Relativistic

$$\left(\frac{\partial}{\partial t} + \underline{v}\cdot\underline{\nabla}\right)\underline{v} = -\frac{1}{\rho}\nabla p - \nabla\phi$$

$$\left(\frac{\partial}{\partial t} + \underline{v}\cdot\underline{\nabla}\right)\rho = -\rho\,(\underline{\nabla}\cdot\underline{v})$$

$$\nabla^2\phi = 4\pi G\rho$$

$$\rho = \rho_0 + \delta\rho\,, \qquad \underline{v} = \underline{v}_0 + \delta\underline{v}$$

$$\boxed{\rho_0 = \rho_0(t) = \bar{\rho}_0\left(\frac{R}{R_0}\right)^{-3}}$$

$$\underline{v}_0 = H(t)\underline{x}$$

$$\left[\frac{\partial}{\partial t} + (\underline{v}_0 + \delta\underline{v})\cdot\underline{\nabla}\right](\rho_0 + \delta\rho) = -(\rho_0 + \delta\rho)\,\underline{\nabla}\cdot(\underline{v}_0 + \delta\underline{v})$$

$$\left[\frac{\partial}{\partial t} + (\underline{v}_0 + \delta\underline{v})\cdot\underline{\nabla}\right](\underline{v}_0 + \delta\underline{v}) = -\frac{1}{\rho_0 + \delta\rho}\nabla(p_0 + p_1) - \underline{\nabla}(\phi_0 + \phi_1)$$

Zeroth order solutions

$$\left(\frac{\partial}{\partial t} + \underline{v}_0\cdot\underline{\nabla}\right)\rho_0 = -\rho_0\,\underline{\nabla}\cdot\underline{v}_0$$

$\rho_0 = \rho_0(t)$

$v_0 = H\underline{x}$

$$\Rightarrow \quad \dot{\rho}_0 = -3H\rho_0$$

$$\frac{D}{Dt} \equiv \frac{\partial}{\partial t} + (\underline{v}\cdot\underline{\nabla})$$

$$\frac{d}{dt} = \frac{\partial}{\partial t} + (v_0\cdot\nabla)$$

Linearization:

$$\left[\frac{\partial}{\partial t} + (v_0\cdot\nabla)\right]\delta\rho = -\rho_0\,\underline{\nabla}\cdot\delta\underline{v} - \delta\rho\,\underbrace{\underline{\nabla}\cdot v_0}_{3H}$$

$$\delta \equiv \frac{\delta\rho}{\rho_0} \qquad \frac{d}{dt}\delta = -\underline{\nabla}\cdot\underline{v}$$

Resulting equations:

$$\frac{d}{dt}\delta = -\underline{\nabla}\cdot\delta\underline{U}$$

$\frac{d}{dt} = \frac{\partial}{\partial t} + (\underline{U}_0\cdot\underline{\nabla})$

$\delta \equiv \frac{\delta\rho}{\rho_0}$

$$\frac{d}{dt}\delta\underline{U} = -\frac{\underline{\nabla}\,\delta P}{\rho_0} - \nabla\,\delta\phi \;\; - \underbrace{(\delta\underline{U}\cdot\underline{\nabla})\,U_0}_{H\,\delta\underline{U}}$$

$$\nabla^2\delta\phi = 4\pi G\rho_0\delta$$

$$\left[(\delta\underline{U}\cdot\underline{\nabla})\,\underline{U}_0\right]_j \longrightarrow \underbrace{\left(\delta U_i \frac{\partial}{\partial X_i}\right) H X_j}_{\delta_{ij}} = H\,\delta U_j$$

Define: $\underline{x}(t) = R(t)\,\underline{r}(t)$

physical coordinate — co-moving coordinate

$$\delta\underline{U} = R(t)\,\underline{u}(t)$$

$$\underline{\nabla}_x = \frac{1}{R(t)}\underline{\nabla}_r$$

$\Longrightarrow$

(1) $\frac{d\delta}{dt} = -\underline{\nabla}_r\cdot\underline{u}$

(2) $\frac{d}{dt}\underline{u} + 2\frac{\dot{R}}{R}u = \frac{g}{R} - \frac{\nabla_r\,\delta P}{R^2}$

$\underline{g} \equiv -\frac{\nabla_r\,\delta\phi}{R}$

$\frac{d}{dt} = \partial_t + \frac{(\underline{U}_0\cdot\nabla_r)}{R}$

Case of $\Omega_0 = 1$, $\Lambda = 0$ and $\lambda >> \lambda_J$

$R(t) \propto t^{2/3}$

$\frac{\dot{R}}{R} = \frac{2}{3} \frac{1}{t}$

$4\pi G \rho_0 = \frac{2}{3} t^2$

$$\frac{d^2}{dt^2} \delta + \frac{4}{3} \frac{1}{t} \frac{d}{dt} \delta = \frac{2}{3 t^2} \delta$$

$\delta = t^\alpha$

$\alpha(\alpha - 1) + \frac{4}{3} \alpha = \frac{2}{3}$

$\alpha = \frac{2}{3}$ or $\alpha = -1$

General solution: $\delta_+(r,t) = \delta_+(r) \left(\frac{t}{t_0}\right)^{2/3}$

$\delta_-(r,t) = \delta_-(r) \left(\frac{t}{t_0}\right)^{-1}$