Initial Conditions

At some early epoch : $z_i >> 1$

$$\delta(x) = \left(\frac{\rho(x)-\bar{\rho}}{\bar{\rho}}\right)_{z_i}$$

$\delta(x)$ is a random field.

$\delta(x)$ is a homogeneous & isotropic random field.

$$\langle \delta(\underline{x}) \rangle = 0$$

$\langle \;\; \rangle$ denotes volume average

cosmology: $\langle \;\; \rangle_{\text{volume}} \equiv \langle \;\; \rangle_{\text{ensemble}}$

Two point correlation function

$\langle \delta(\underline{x})\, \delta(\underline{x}+\underline{r}) \rangle_x$ depends on $\underline{r}$ only

$$\xi(\underline{r}) = \langle \delta(\underline{x})\, \delta(\underline{x}+\underline{r}) \rangle$$

Homogeniety & isotropy $\Rightarrow$ $\xi(\underline{r}) = \xi(r)$

②

$$\xi(\underline{r}) = \langle \delta(\underline{x}+\underline{r})\,\delta(\underline{x}) \rangle_x =$$

$$\frac{1}{V}\int d^3x \;\; \delta(\underline{x}+\underline{r})\,\delta(\underline{x}) = \frac{1}{V}\int d^3x\, d^3k\, d^3k'$$

$$\delta_k\,\delta_{k'}\;\exp[-i(\underline{k}\cdot(x+\underline{r}) + \underline{k}'\underline{x})]$$

$$\frac{1}{V}\int d^3x\, \exp[-i(\underline{k}+\underline{k}')\underline{x}] = \delta_D(\underline{k}+\underline{k}') \quad , \quad \underline{k}' \to -\underline{k}$$

$$\xi(\underline{r}) = \int d^3k\; e^{i\underline{k}\cdot\underline{r}}\; |\delta_k|^2$$

Alternatively:

$$\xi(\underline{r}) = \langle \delta(x+r)\,\delta(x) \rangle_{\text{ensemble}}$$

$$= \int d^3k\, d^3k' \;\; \langle \delta_k \delta_{k'} \rangle\; e^{-i\underline{k}(\underline{x}+\underline{r})}\, e^{-ik'\underline{x}}$$

$$\langle \delta_k \;\; \delta_{k'} \rangle = P(k)\;\delta_D(\underline{k}+\underline{k}')$$

$$\xi(\underline{r}) = \int d^3k \;\; P(k)\, e^{-i\underline{k}\cdot\underline{r}}$$

$$\langle \delta_k \;\; \delta_{k'} \rangle = \int d^3x'\, d^3x\; e^{-i(\underline{k}\cdot\underline{x} + \underline{k}'\underline{x}')} \langle \delta_1(x')\,\delta(x) \rangle$$

$$= \int d^3x\, d^3r\; e^{-i(\underline{k}+\underline{k}')x}\; e^{-i\underline{k}\cdot\underline{r}}\; \xi(r)$$

$$= \delta_D(\underline{k}+\underline{k}')\; \underbrace{\int d^3r\, e^{-i\underline{k}\cdot\underline{r}}\; \xi(r)}_{P(k)}$$

$P(\underline{k})$ & $\xi(\underline{r})$ are Fourier conjugates.

H & I $\longrightarrow$ $P(\underline{k}) = P(k)$

$\xi(\underline{r}) = \xi(r)$

$$\boxed{\begin{aligned} \xi(r) &= \frac{1}{2\pi^2} \int P(k) \frac{\sin kr}{kr} k^2 dk \\ \xi(\underline{r}) &= \frac{1}{(2\pi)^3} \int P(\underline{k})\, e^{-i\underline{k}\cdot\underline{r}}\, d^3k \end{aligned}}$$

$\delta(r)$ is a Gaussian random field

$$P[\delta(x)] = \frac{1}{(2\pi)^{1/2}} \exp\left[-\frac{\delta^2_{(x)}}{2\sigma_0^2}\right]$$

PDF $\uparrow$

Joint probability of n points: $\delta(x_1), \delta(x_2) \ldots \delta(x_n)$

~~$P[\delta(x_1), \delta(x_2)] = \frac{1}{2\pi} \exp[-(\delta_1 \delta_2)]$~~

$$P[\delta_1, \ldots \delta_n] = \frac{1}{(2\pi)^{n/2} |R|^{1/2}} \exp[-\tfrac{1}{2}\underline{\delta}\, R^{-1}\, \underline{\delta}^{T}]$$

where R is the correlation matrix

$$R_{ij} \equiv \langle \delta_i \delta_j \rangle = \langle \delta(x_i)\, \delta(x_j) \rangle = \xi(r_{ij})$$

$$r_{ij} = |\underline{r}_i - \underline{r}_j|$$

How to relate the powerspectrum to the final structure?

Suppose that $P(k) \propto k^n$ $\quad -3 < n < 1$

power on all scales:

Suppose one is interested in fluctuations on scales M (in terms of co-moving scales R, $R = \left(\frac{M}{4\pi\rho}\right)^{1/3} \propto M^{1/3}$)

Filter (or smooth) all fluctuations on scales $r \leq R$:

$$\bar{\delta}_f(x) \equiv \int d^3r \, \delta(x+r) W_f(r)$$

filter

filter (or window) function

For example: a Gaussian filter

$$W_f(r) \propto \exp\left[-\frac{1}{2}\left(\frac{r}{R}\right)^2\right]$$

In k space: $\quad W_f(r) = e^{-\frac{r^2}{2R^2}} \rightarrow W_k^f = e^{-\frac{k^2R^2}{2}}$

$$\delta_k^f = \delta_k e^{-\frac{k^2R^2}{2}}$$

$$\sigma_f^2 \equiv \langle \delta_f^2 \rangle = \langle \int d^3k\, d^3k'\; \delta_k e^{-\frac{k^2}{2}R^2} e^{-i\underline{k}\cdot\underline{x}}$$

$$\delta_{k'} e^{-\frac{k'^2 R^2}{2}} e^{-i\underline{k}'\cdot\underline{x}} \rangle =$$

$$= \int d^3k\, d^3k'\; P(k)\, \delta_D(\underline{k}+\underline{k}')\, e^{-\frac{k^2+k'^2}{2R^2}}\, e^{-i(\underline{k}+\underline{k}')x}$$

$$\boxed{\langle \delta_k \delta_k' \rangle = P(k)\, \delta_D(\underline{k}+\underline{k}')} \quad = \int d^3k\; P(k)\, e^{-\frac{k^2}{R^2}}$$

For $P(k) \propto k^n$

$$\sigma_f^2 \propto R^{-(n+3)}$$

$$\propto M^{-\frac{n+3}{3}}$$

Now, one can use different filtering.

Top-hat window: $W_{TH}(r) = \begin{cases} 1 & r < R \\ 0 & r > R \end{cases}$

Again for $P(k) \propto k^n$ $\quad \sigma_{TH}(M) = \text{const}\; M^{-\frac{n+3}{6}}$

(6)

Crude cosmology:

(1) Fix scale of interest ($M = 10^{12}$ or 10^{15} $M_\odot$)
$\uparrow$ galaxy $\uparrow$ cluster

(2) For a given $P(k) \longrightarrow \sigma_M$

(3) Choose $\delta_M(x) = \nu\, \sigma_M$

fluctuation on a scale M at position X

amplitude in r.m.s. units

$\nu = 1, 2, 3$ say

(4) Assume spherical TH fluctuation

(5) Use the non-linear TH model to calculate R_{max} ($\frac{R_{max}}{R_i} \propto \delta_M^{-1}$), collapse time, virial density and so on