Jeans Instability

$$\tau_{pressure} \sim \frac{\lambda}{c_s}$$

$$c_s^2 = \left(\frac{\partial P}{\partial \rho}\right)_S$$

$$\tau_{grav} \sim \frac{1}{\sqrt{G\rho}}$$

$\lambda > \lambda_J = \frac{c_s}{\sqrt{G\rho}}$ gravitational instability

$\lambda < \lambda_J$ ~~damping~~ stability: sound waves

Continuity eq. $\frac{\partial \rho}{\partial t} + \underline{\nabla}(\rho \cdot \underline{v}) = 0$

Euler eq. $\underbrace{\frac{\partial}{\partial t}\underline{v} + (\underline{v}\cdot\underline{\nabla})\underline{v}}_{\text{acceleration}} = -\underbrace{\frac{1}{\rho}\underline{\nabla}P}_{\text{pressure gradient}} - \underbrace{\nabla\phi}_{\text{grav. acc.}}$

Small perturbation: $\rho = \rho_0 + \rho_1 + \cdots$

$P_0, \rho_0 = \text{const.}$ $\quad P = P_0 + P_1 + \cdots$

In general one needs an equation of state:

$P = P(\rho, T)$ (say)

We assume adiabatic perturbations

$\left(\frac{\partial P}{\partial \rho}\right)_S \equiv v_s^2$ ← sound speed

$P_1 = v_s^2 \rho_1$ (to 1st order)

$v_s^* = v_s(\rho_0, P_0)$

Linearization of E.O.M:

$$① \quad \frac{\partial \rho_1}{\partial t} + \rho_0 \nabla \cdot \underline{v}_1 = 0$$

$$② \quad \frac{\partial \underline{v}_1}{\partial t} = -\frac{v_s^2}{\rho_0} \nabla \rho_1 - \nabla \phi_1$$

$$\nabla^2 \phi_1 = 4\pi G \rho_1$$

$$\left.\begin{array}{l} \frac{\partial}{\partial t} : ① \\ \nabla \cdot ② \end{array}\right\} \Longrightarrow \quad \frac{\partial^2}{\partial t^2} \rho_1 - v_s^2 \nabla^2 \rho_1 = 4\pi G \rho_0 \rho_1$$

$$\rho_1(x,t) = \sum_{k,\omega} A(k,\omega) \exp[-i(\underline{k}\cdot\underline{r} - \omega t)]$$

Dispersion relation:

$$\omega^2 = v_s^2 k^2 - 4\pi G \rho_0$$

$\omega^2 > 0$ stability

$\omega^2 < 0$ instability

$$v_s^2 k_J^2 = 4\pi G \rho_0$$

$$\left\{\begin{array}{l} k_J \equiv \left(\frac{4\pi G \rho_0}{v_s^2}\right)^{1/2} \\ \lambda_J \equiv \left(\frac{\pi}{G\rho_0}\right)^{1/2} v_s \end{array}\right.$$

$$c_s^2 \equiv \left(\frac{\partial P}{\partial \rho}\right)_s$$

$$\delta(r,t) = \sum \delta_k(t)\, e^{-i\mathbf{k}\cdot\mathbf{r}}$$

$$\nabla\cdot (2),\ \frac{\partial}{\partial t}(1) \Longrightarrow \boxed{\ddot{\delta} + 2\frac{\dot{R}}{R}\dot{\delta} = \delta\left(4\pi G\rho_0 - \frac{c_s^2 k^2}{R^2}\right)}$$

Jeans scale appears also in the ~~expande~~d (co-moving) case.

Equation of motions for particles:

$$\underline{X} = R(t)\ \underline{r}$$

$$\dot{X} = \frac{\dot{R}}{R} X + R\dot{r} = \underbrace{H(t)\,\underline{X}}_{\text{Hubble expansion}} + R u$$

$$\ddot{X} = \underbrace{R\dot{u} + 2\dot{R}u}_{\underline{g}} + \underbrace{\frac{\ddot{R}}{R}X}_{g_0} = \underline{g} + \underline{g}_0$$

Linear Perturbation Theory: Non-Relativistic

$$\left(\frac{\partial}{\partial t} + \underline{v}\cdot\underline{\nabla}\right)\underline{v} = -\frac{1}{\rho}\underline{\nabla}P - \underline{\nabla}\phi$$

$$\left(\frac{\partial}{\partial t} + \underline{v}\cdot\underline{\nabla}\right)\rho = -\rho\,(\underline{\nabla}\cdot\underline{v})$$

$$\nabla^2\phi = 4\pi G\rho$$

$$\rho = \rho_0 + \delta\rho\,, \qquad \underline{v} = \underline{v}_0 + \delta\underline{v}$$

$$\boxed{\rho_0 = \rho_0(t) = \bar{\rho}_0\left(\frac{R}{R_0}\right)^{-3}}$$

$$\underline{v}_0 = H(t)\underline{x}$$

$$\left[\frac{\partial}{\partial t} + (\underline{v}_0 + \delta\underline{v})\cdot\underline{\nabla}\right](\rho_0 + \delta\rho) = -(\rho_0 + \delta\rho)\,\underline{\nabla}\cdot(\underline{v}_0 + \delta\underline{v})$$

$$\left[\frac{\partial}{\partial t} + (\underline{v}_0 + \delta\underline{v})\cdot\underline{\nabla}\right](\underline{v}_0 + \delta\underline{v}) = -\frac{1}{\rho_0 + \delta\rho}\nabla(P_0 + P_1) - \underline{\nabla}(\phi_0 + \phi_1)$$

Zeroth order solutions

$$\left(\frac{\partial}{\partial t} + \underline{v}_0\cdot\underline{\nabla}\right)\rho_0 = -\rho_0\,\underline{\nabla}\cdot\underline{v}_0$$

$\rho_0 = \rho_0(t)$

$v_0 = H\,\underline{x}$

$$\Rightarrow \qquad \dot{\rho}_0 = -3H\rho_0$$

$$\frac{D}{Dt} \equiv \frac{\partial}{\partial t} + (\underline{v}\cdot\underline{\nabla})$$

$$\frac{d}{dt} = \frac{\partial}{\partial t} + (v_0\cdot\underline{\nabla})$$

Linearization:

$$\left[\frac{\partial}{\partial t} + (v_0\cdot\underline{\nabla})\right]\delta\rho = -\rho_0\,\underline{\nabla}\cdot\delta\underline{v} - \delta\rho\,\underbrace{\underline{\nabla}\cdot\underline{v}_0}_{3H}$$

$$\delta \equiv \frac{\delta\rho}{\rho_0} \qquad \frac{d}{dt}\delta = -\underline{\nabla}\cdot\underline{v}$$

Resulting equations:

$$\frac{d}{dt}\delta = -\underline{\nabla}\cdot\delta\underline{U}$$

$\frac{d}{dt} = \frac{\partial}{\partial t} + (U_0\cdot\underline{\nabla})$

$\delta = \frac{\delta\rho}{\rho_0}$

$$\frac{d}{dt}\delta\underline{U} = -\frac{\underline{\nabla}\,\delta p}{\rho_0} - \nabla\delta\phi \underbrace{-(\delta\underline{U}\cdot\underline{\nabla})U_0}_{H\,\delta\underline{U}}$$

$$\nabla^2\delta\phi = 4\pi G\rho_0\delta$$

$$\left[(\delta\underline{U}\cdot\underline{\nabla})\,\underline{U_0}\right]_j \longrightarrow \underbrace{\left(\delta U_i\,\frac{\partial}{\partial X_i}\right) H X_j}_{\delta_{ij}} = H\,\delta U_j$$

Define: $\underline{x}(t) = R(t)\,\underline{r}(t)$

physical coordinate, co-moving coordinate

$$\delta\underline{U} = R(t)\,\underline{u}(t)$$

$$\underline{\nabla}_x = \frac{1}{R(t)}\underline{\nabla}_r$$

$\Longrightarrow$

$$\boxed{\begin{aligned} &(1)\quad \frac{d}{dt}\delta = -\underline{\nabla}_r\cdot\underline{u} \\ &(2)\quad \frac{d}{dt}\underline{u} + 2\frac{\dot R}{R}\underline{u} = \frac{\underline{g}}{R} - \frac{\nabla_r\,\delta p}{R^2} \\ &\underline{g} \equiv -\frac{\nabla_r\,\delta\phi}{R} \\ &\frac{d}{dt} = \partial_t + \frac{(\underline{U_0}\cdot\nabla_r)}{R} \end{aligned}}$$

Case of $\Omega_0 = 1$, $\Lambda = 0$ and $\lambda \gg \lambda_J$

$R(t) \propto t^{2/3}$

$\frac{\dot{R}}{R} = \frac{2}{3} \frac{1}{t}$

$4\pi G \rho_0 = \frac{2}{3} t^2$

$$\frac{d^2}{dt^2}\delta + \frac{4}{3}\frac{1}{t}\frac{d}{dt}\delta = \frac{2}{3}t^2 \delta$$

$\delta = t^\alpha$

$\alpha(\alpha - 1) + \frac{4}{3}\alpha = \frac{2}{3}$

$\alpha = \frac{2}{3}$ or $\alpha = -1$

General solution: $\delta_+(r,t) = \delta_+(r)\left(\frac{t}{t_0}\right)^{2/3}$

$\delta_-(r,t) = \delta_-(r)\left(\frac{t}{t_0}\right)^{-1}$

Peculiar Velocities

$$\frac{\partial}{\partial t}\delta = -\underline{\nabla}\cdot\underline{u}$$

$$\frac{\partial}{\partial t}u + 2\frac{\dot{R}}{R}u = -\frac{\underline{g}}{R}$$

$$-\underline{\nabla}\cdot\underline{g} = 4\pi G\rho_0 R\delta$$

$$\left[\begin{array}{l}\underline{\nabla}\to\nabla_r \\ \underline{U} = Ru \\ \underline{g}(r,t) = G\rho_0 R\int d^3r'\,\delta(r',t)\dfrac{(\underline{r}-\underline{r}')}{|\underline{r}-\underline{r}'|^3}\end{array}\right]$$

$$\frac{\partial}{\partial t}\left(\frac{\underline{\nabla}\cdot\underline{g}}{4\pi G\rho_0 R}\right) = -\underline{\nabla}\cdot\underline{u}$$

$$\underline{u} = +\frac{\partial}{\partial t}\left(\frac{\underline{g}}{4\pi G\rho_0 R}\right) + \underbrace{\underline{F}(r,t)}$$

$\underline{\nabla}\cdot\underline{F} = 0$ decaying mode

Analysis applying to the growing mode:

$$\underline{g}(r,t) = G\rho_0 R\underbrace{D(t)}_{\text{growing mode}}\underbrace{\int d^3r'\,\delta(r')\frac{(\underline{r}-\underline{r}')}{|\underline{r}-\underline{r}'|^3}}_{\tilde{g}}$$

$$\underline{u} = +\frac{\tilde{\underline{g}}}{4\pi}\frac{d}{dt}D = -\frac{\tilde{\underline{g}}}{4\pi D(t)}\dot{D}$$

$$\underline{u} = +\frac{\underline{g}}{4\pi G\rho_0 R}\frac{1}{D}\frac{dD}{dt}$$

$$\Rightarrow \quad \underline{u} = +\frac{H f(\Omega)\,\underline{g}}{4\pi G\rho_0 R}, \quad f(\Omega) = \frac{R}{D}\frac{dD}{dR} \sim \Omega^{0.6}$$

$$\underline{u}(r,t) = \frac{H}{4\pi} f(\Omega) \int d^3r' \, \delta(r') \frac{(\underline{r}-\underline{r}')}{|\underline{r}-\underline{r}'|^3}$$

Or:

$$\underline{u}(\underline{r},t) = \frac{2}{3} \frac{f(\Omega)\, \underline{g}}{H\, \Omega R}$$

$$\delta \underline{v}(r,t) = \frac{2}{3} \frac{f(\Omega)\, \underline{g}}{H\, \Omega}$$

Case of $\underline{\nabla}\cdot\underline{u} = 0$, $\underline{\nabla}\times\underline{u} \neq 0$

$$\frac{\partial}{\partial t} u + 2 \frac{\dot{R}}{R} u = 0$$

$$u(r,t) = f(r)\, t^{\alpha}$$

$$\alpha \frac{u}{t} + 2 \cdot \frac{2}{3} \frac{u}{t} = 0$$

$$\alpha = -\frac{4}{3}$$

$$u \propto t^{-4/3}$$

$$\Rightarrow v \propto t^{-2/3} \propto \frac{1}{R}$$

Density Perturbation in an open universe:

$\Omega_0 < 1$: for $z \gg 1$ $\quad R \propto t^{2/3}$

The universe behaves like an Einstein-de Sitter model:

At $z \gg 1$ expect $\delta \propto D_{\Omega_0 = 1}(t) \propto R(t) \propto t^{1/2}$

Suppose that at some initial time, $z_i \gg 1$, the growing mode already dominates and $\delta(z_i) \sim \delta_i$.

It can be shown that

$$\frac{\delta(t \to \infty)}{\delta_i} = \frac{1}{\Omega_i^{-1} - 1}$$

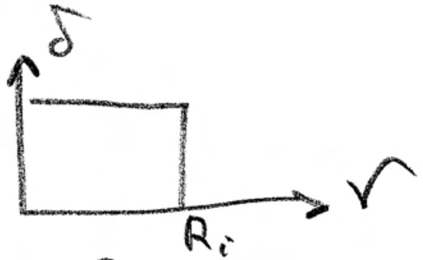
Namely there is a time after whic δ FREEZES and stops to grow.

$$z_{freeze} \sim \Omega_i^{-1} - 1$$

(Note $\Omega_0 = 1 \rightarrow z_{freeze} = -1 \; (\Leftrightarrow t = \infty)$)

Spherical Top-hat Model: Non-linear Evolution

Consider a spherical density perturbation:



Because of the spherical symmetry one can consider the perturbation to be a mini-Friedman universe

Equations of motion $\Rightarrow R(t) = A(1-\cos\theta)$ $\qquad A^3 = GMB^2$

$$t = B(\theta - \sin\theta)$$

The perturbation reaches max expansion

at $\theta = \pi \Rightarrow R_{max} = 2A$

$$t = \pi B$$

(2) $A = (R_{max}/2)$

$$B = \left(\frac{GM}{(R_{max}/2)^3}\right)^{-1/2}$$

$t_{max} \Rightarrow$

$$t_{max} = \pi \frac{1}{\sqrt{\frac{GM}{(R_{max}/2)}}}$$

3

$$\frac{R_{max}}{R_i} = \frac{1+\delta_i}{\delta_i - (\Omega_i^{-1}-1)} \sim \frac{1}{\delta_i - (\Omega_i^{-1}-1)}$$

$Z_i \gg 1$

$\Omega(Z_i) \sim 1$

For $\Omega_0 = 1$ $\qquad \frac{R_{max}}{R_i} = \delta_i^{-1}$

Virialization: Collapse $t_{coll} = 2\, t_{max}$

$R(t_{coll}) \equiv 0$

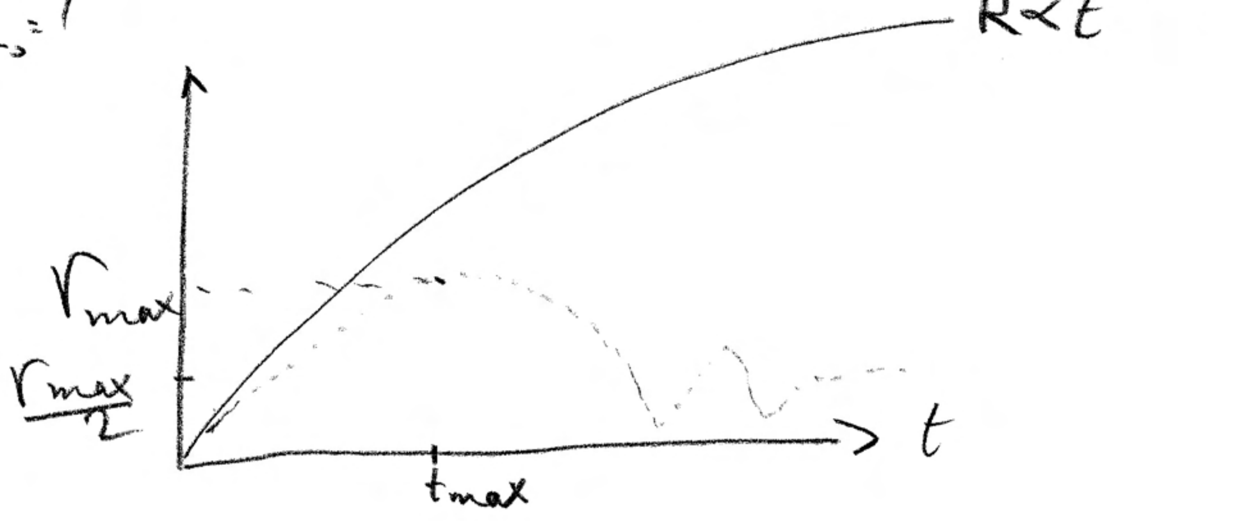
mathematically

At $t \sim t_{coll}$ virialization:

$$R_{vir} \sim \frac{R_{max}}{2}$$

$$t_{vir} \sim t_{coll} \propto \delta_i^{-3/2}$$

$\Omega_0 = 1$



$\Omega_0 = 1.0$

$$\delta_{NL}(t_{max}) = \frac{9\pi^2}{16} \sim 5.5$$

$$\delta_L(t_{max}) = \frac{3}{20}\left(6\pi^2\right)^{2/3} \sim 1.06$$

$$1+\delta_{vir} \sim 178\,\Omega_0^{-0.7}$$

Initial Conditions ①

At some early epoch: $z_i >> 1$

$$\delta(x) = \left(\frac{\rho(x)-\bar{\rho}}{\bar{\rho}}\right)_{z_i}$$

$\delta(x)$ is a random field

$\delta(x)$ is a homogeneous & isotropic random field.

$$\langle \delta(\underline{x}) \rangle = 0$$

$\langle \ \rangle$ denotes volume average

cosmology: $\langle \ \rangle_{\text{volume}} \equiv \langle \ \rangle_{\text{ensemble}}$

Two point correlation function

$\langle \delta(\underline{x})\, \delta(\underline{x}+\underline{r}) \rangle_x$ depends on $\underline{r}$ only

$$\xi(\underline{r}) = \langle \delta(\underline{x})\, \delta(\underline{x}+\underline{r}) \rangle$$

Homogeniety & isotropy $\Rightarrow$ $\xi(\underline{r}) = \xi(r)$

$$\xi(\underline{r}) = \langle \delta(\underline{x}+\underline{r})\, \delta(\underline{x}) \rangle_x =$$

$$\frac{1}{V}\int d^3x \;\; \delta(\underline{x}+\underline{r})\, \delta(\underline{x}) = \frac{1}{V}\int d^3x\, d^3k\, d^3k'$$

$$\delta_k\, \delta_{k'}\; \exp[-i(\underline{k}\cdot(x+\underline{r}) + \underline{k}'\underline{x})]$$

$$\frac{1}{V}\int d^3x \exp[-i(\underline{k}+\underline{k}')\underline{x}] = \delta_D(\underline{k}+\underline{k}') \quad , \quad \underline{k}' \to -\underline{k}$$

$$\xi(\underline{r}) = \int d^3k\; e^{i\underline{k}\cdot\underline{r}}\; |\delta_k|^2$$

Alternatively:

$$\xi(\underline{r}) = \langle \delta(x+r)\, \delta(x) \rangle_{ensemble}$$

$$= \int d^3k\, d^3k' \;\; \langle \delta_k \delta_{k'} \rangle\, e^{i\underline{k}(\underline{x}+\underline{r})}\, e^{-ik'\underline{x}}$$

$$\langle \delta_k \;\delta_{k'} \rangle = P(k)\;\delta_D(\underline{k}+\underline{k}')$$

$$\xi(\underline{r}) = \int d^3k\;\; P(k)\, e^{-i\underline{k}\cdot\underline{r}}$$

$$\langle \delta_k \;\; \delta_{k'} \rangle = \int d^3x'\, d^3x\; e^{-i(\underline{k}\cdot\underline{x} + \underline{k}'\underline{x}')} \langle \delta(x')\delta(x)\rangle$$

$$= \int d^3x\, d^3r\; e^{-i(\underline{k}+\underline{k}')x}\; e^{-i\underline{k}\underline{r}}\;\; \xi(r)$$

$$= \delta_D(\underline{k}+\underline{k}')\;\; \underbrace{\int d^3r\, e^{-i\underline{k}\cdot\underline{r}}\; \xi(r)}_{P(k)}$$

$p(\underline{k})$ & $\xi(\underline{r})$ are Fourier conjugates.

H & I $\longrightarrow$ $P(\underline{k}) = P(k)$

$\xi(\underline{r}) = \xi(r)$

$$\xi(r) = \frac{1}{2\pi^2} \int p(k) \frac{\sin kr}{kr} k^2 dk$$

$$\xi(\underline{r}) = \frac{1}{(2\pi)^3} \int p(\underline{k}) e^{-i\underline{k}\cdot\underline{r}} d^3k$$

$\delta(r)$ is a Gaussian random field

$$P[\delta(x)] = \frac{1}{(2\pi)^{1/2}} \exp\left[-\frac{\delta^2_{(x)}}{2\sigma_0^2}\right]$$

PDF ↑

Joint probability of n points: $\delta(x_1), \delta(x_2) \ldots \delta(x_n)$

$$\cancel{P[\delta(x_1), \delta(x_2)] = \frac{1}{2\pi} \exp\left[-(\delta_1 \delta_2)\right]}$$

$$P[\delta_1, \ldots \delta_n] = \frac{1}{(2\pi)^{n/2} |R|^{1/2}} \exp\left[-\tfrac{1}{2}\underline{\delta}\, R^{-1}\, \underline{\delta}^{T}\right]$$

where R is the correlation matrix

$$R_{ij} \equiv \langle \delta_i \delta_j \rangle = \langle \delta(x_i)\, \delta(x_j) \rangle = \xi(r_{ij})$$

$r_{ij} = |\underline{r}_i - \underline{r}_j|$

How to relate the powerspectrum to the final structure?

Suppose that $P(k) \propto k^n$ $\quad -3 < n < 1$

power on all scales:

Suppose one is interested in fluctuations on scales M (in terms of co-moving scales R, $R = \left(\frac{M}{4\pi\bar{\rho}}\right)^{1/3} \propto M^{1/3}$)

Filter (or smooth) all fluctuations on scales $r \leq R$:

$$\bar{\delta}_f(x) \equiv \int d^3r \, \delta(x+r) W_f(r)$$

filter

filter (or window) function

For example: a Gaussian filter

$$W_f(r) \propto \exp\left[-\frac{1}{2}\left(\frac{r}{R}\right)^2\right]$$

In k space: $W_f(r) = e^{-\frac{r^2}{2R^2}} \rightarrow W_k^f = e^{-\frac{k^2R^2}{2}}$

$$\delta_k^f = \delta_k e^{-\frac{k^2R^2}{2}}$$

$$\sigma_f^2 \equiv \langle \delta_f^2 \rangle = \langle \int d^3k\, d^3k' \;\delta_k e^{-\frac{k^2}{2}R^2} e^{-i\underline{k}\cdot\underline{x}}$$

$$\delta_{k'} e^{-\frac{k'^2 R^2}{2}} e^{-i\underline{k}'\cdot\underline{x}} \rangle =$$

$$= \int d^3k\, d^3k' \;P(k)\, \delta_D(\underline{k}+\underline{k}')\, e^{-\frac{k^2+k'^2}{2R^2}}\, e^{-i(\underline{k}+\underline{k}')x}$$

$$\boxed{\langle \delta_k \delta_{k'} \rangle = P(k)\, \delta_D(\underline{k}+\underline{k}')} \quad = \int d^3k\; P(k)\, e^{-\frac{k^2}{R^2}}$$

For $P(k) \propto k^n$

$$\sigma_f^2 \propto R^{-(n+3)}$$

$$\propto M^{-\frac{n+3}{3}}$$

Now, one can use different filtering.

Top-hat window: $W_{TH}(r) = \begin{cases} 1 & r < R \\ 0 & r > R \end{cases}$

Again for $P(k) \propto k^n$ $\quad \sigma_{TH}(M) = \text{const}\; M^{-\frac{n+3}{6}}$

Crude cosmology:

1. Fix scale of interest ($M = 10^{12}$ or $10^{15}\ M_\odot$)
   galaxy ↑ cluster ↑

2. For a given $P(k) \longrightarrow \sigma_M$

3. Choose $\delta_M(x) = \nu\,\sigma_M$

   fluctuation on a scale M at position X

   amplitude in r.m.s. units

   $\nu = 1, 2, 3$ say

4. Assume spherical TH fluctuation.

5. Use the non-linear TH model to calculate R_{max} $\left(\frac{R_{max}}{R_i} \propto \delta_M^{-1}\right)$, collapse time, virial density and so on

The Evolution of the Jeans Mass

Z_{eq} = equality of matter & radiation densities: $\rho_m(z_{eq}) \equiv \rho_R(z_{eq})$

non-relativistic

$Z \gg Z_{eq}$: $\quad V_S = \left(\frac{\partial P}{\partial \rho}\right)_S^{1/2}$, $\quad P = P_R + \underset{\overset{\parallel}{0}}{P_M} \sim P_R \sim \frac{\rho_R c^2}{3}$

$$V_S \sim \frac{c}{\sqrt{3}}$$

Consider baryons: $Z_{rec} < Z < Z_{eq}$

$$\left(\Omega_b h^2 > 4 \times 10^{-2}\right)$$

During that period $\quad Z_{rec} < Z \ll Z_{eq}$

$$\rho \sim \rho_M, \quad P \sim P_R$$

$$V_S = \frac{c}{\sqrt{3}} \, \frac{4}{3}\left(\frac{\rho_R}{\rho_M}\right)^{1/2} \sim \frac{c}{\sqrt{3}}\left(\frac{1+z}{1+z_{eq}}\right)^{1/2}$$

$$\sim 2 \times 10^{8}\left(\frac{1+z}{1+z_{eq}}\right)^{1/2} \frac{m}{s}$$

$$\cancel{\lambda_J = V_s \sqrt{\frac{\pi}{G\rho}}}$$

$$\cancel{M_J \equiv \frac{4\pi}{3}\rho_M\left(\frac{\lambda_J}{2}\right)^3}$$

For $z < z_{rec}$:

We have shown that $v \propto R^{-1}$

velocity perturbation
without density perturbation

$$\Rightarrow T_M \propto v^2 \propto R^{-2} \propto (1+z)^2$$

particles

$$T_R \propto R^{-1} \propto (1+z)$$

$$P \propto \rho^{\gamma}$$

$$P = n k_B T$$

$$v_s = \left(\frac{\partial P_M}{\partial \rho_M}\right)_S^{1/2}$$

$$= \left(\gamma \frac{k_B T}{m_p}\right)^{1/2}$$

Reminder: $\lambda_J = \left(\frac{\pi}{G\rho}\right)^{1/2} v_s$

$$M_J \equiv \frac{4\pi}{3} \rho_M \left(\frac{\lambda_J}{2}\right)^3$$

Reminder: $\rho(z \gg z_{eq}) \sim \rho_R = \rho_R(z_{eq}) \frac{(1+z)^4}{(1+z_{eq})^4}$ ③

$$\rho(z \ll z_{eq}) \sim \rho_M = \rho_R(z_{eq}) \left(\frac{1+z}{1+z_{eq}}\right)^3$$

Show that: $M_J(z \gg z_{eq}) \sim M_J(z_{eq}) \left(\frac{1+z}{1+z_{eq}}\right)^{-3}$

$$M_J(z_{eq}) \sim 3.5 \times 10^{15} (\Omega h^2)^{-2} M_\odot$$

For $z_{rec} < z < z_{eq}$:

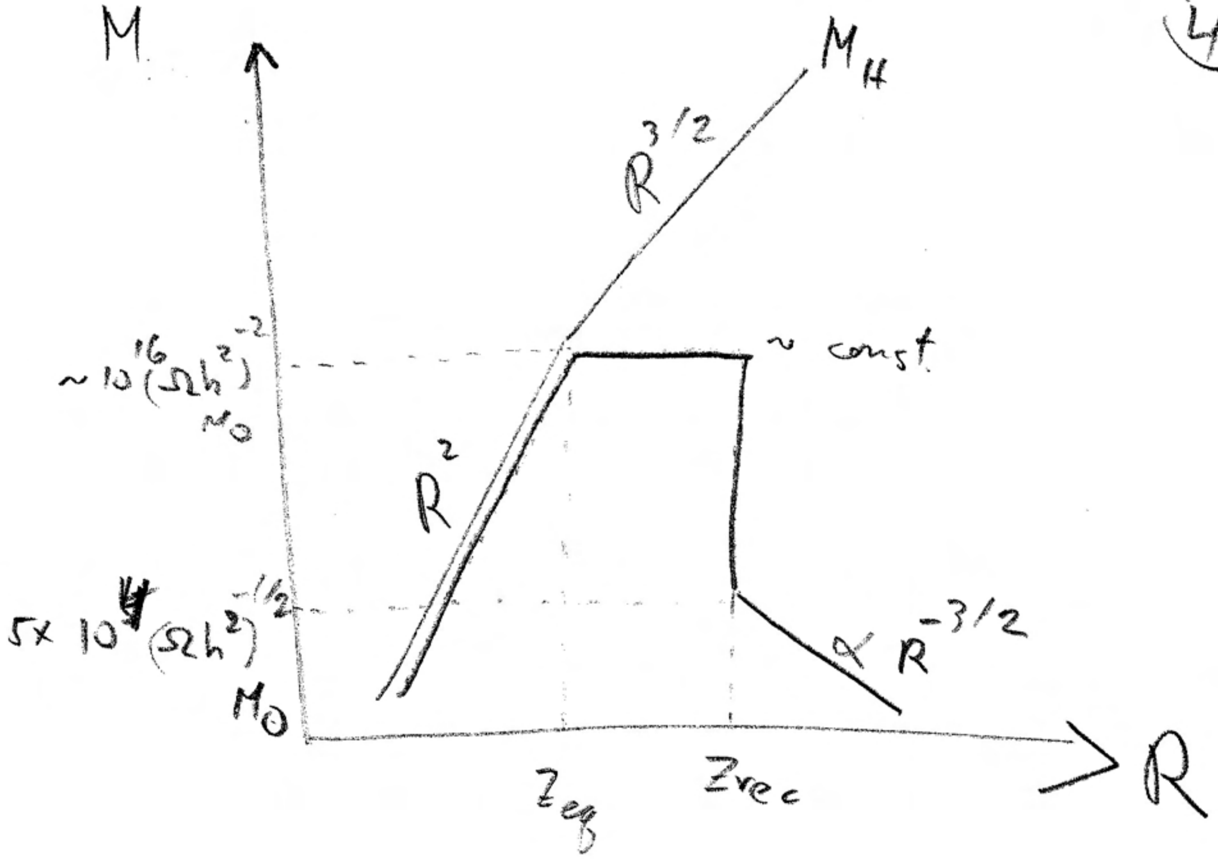
$$M_J(z) \sim \frac{\pi}{6} \rho_m \left[\frac{c}{\sqrt{3}} \left(\frac{1+z}{1+z_{eq}}\right)^{1/2} \left(\frac{\pi}{G\rho}\right)^{1/2} \right]^{1/3}$$

$$\sim M_J(z_{eq}) \sim \text{const}$$

For: $M_J(z < z_{rec}) \sim \frac{\pi}{6} \rho_m \left(\frac{\pi k_B T_M}{G m_p \rho_m} \right)^{3/2}$

$$\sim M_J(z_{rec}) \left(\frac{1+z}{1+z_{rec}}\right)^{3/2}$$

$T_{rec} \sim 1400°K$


M
M_H
$R^{3/2}$
$\sim 10^{16}(\Omega h^2)^{-2}$
$M_\odot$
$\sim$ const
R^2
$5\times 10^{4}(\Omega h^2)^{-1/2}$
$M_\odot$
$\propto R^{-3/2}$
z_{eq}
z_{rec}
R

$$t_{cool} \equiv \frac{E}{\dot{E}} \sim \frac{\frac{3}{2}\rho k_B T}{\mu \, \Lambda(T)}$$

$$t_{dyn} \sim \frac{\pi}{2}\left[\frac{2GM}{R^3}\right]^{-1/2}$$

$$t_{cool} \sim 8\times10^6\,\mathrm{yr}\left(\frac{n}{1\,\mathrm{cm}^{-3}}\right)^{-1} \cdot \left[\left(\frac{T}{10^6\,\mathrm{K}}\right)^{-1/2} + 1.5 f_m\left(\frac{T}{10^6}\right)^{-3/2}\right]^{-1}$$

f_m ↑ Bremsstrahlung f-f ← lines f-b b-b

$f_m \sim 1$ primordial, no metals

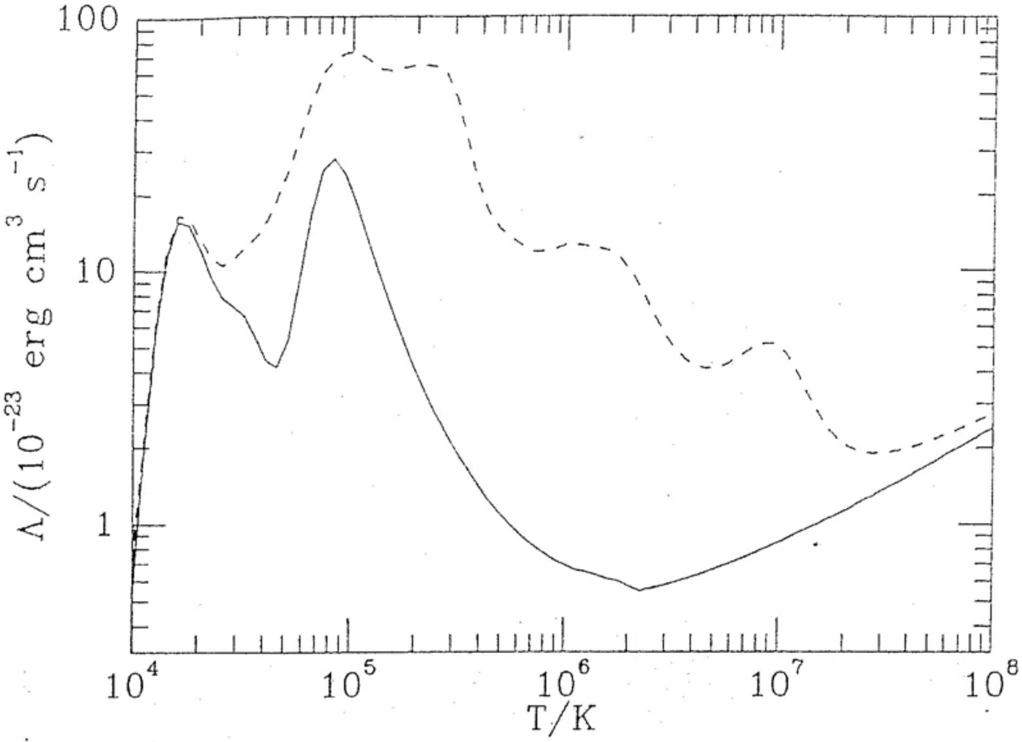
$f_m \sim 30$ solar abundance

For $T < 10^6$, $f = 1$, $\mu = 0.57$

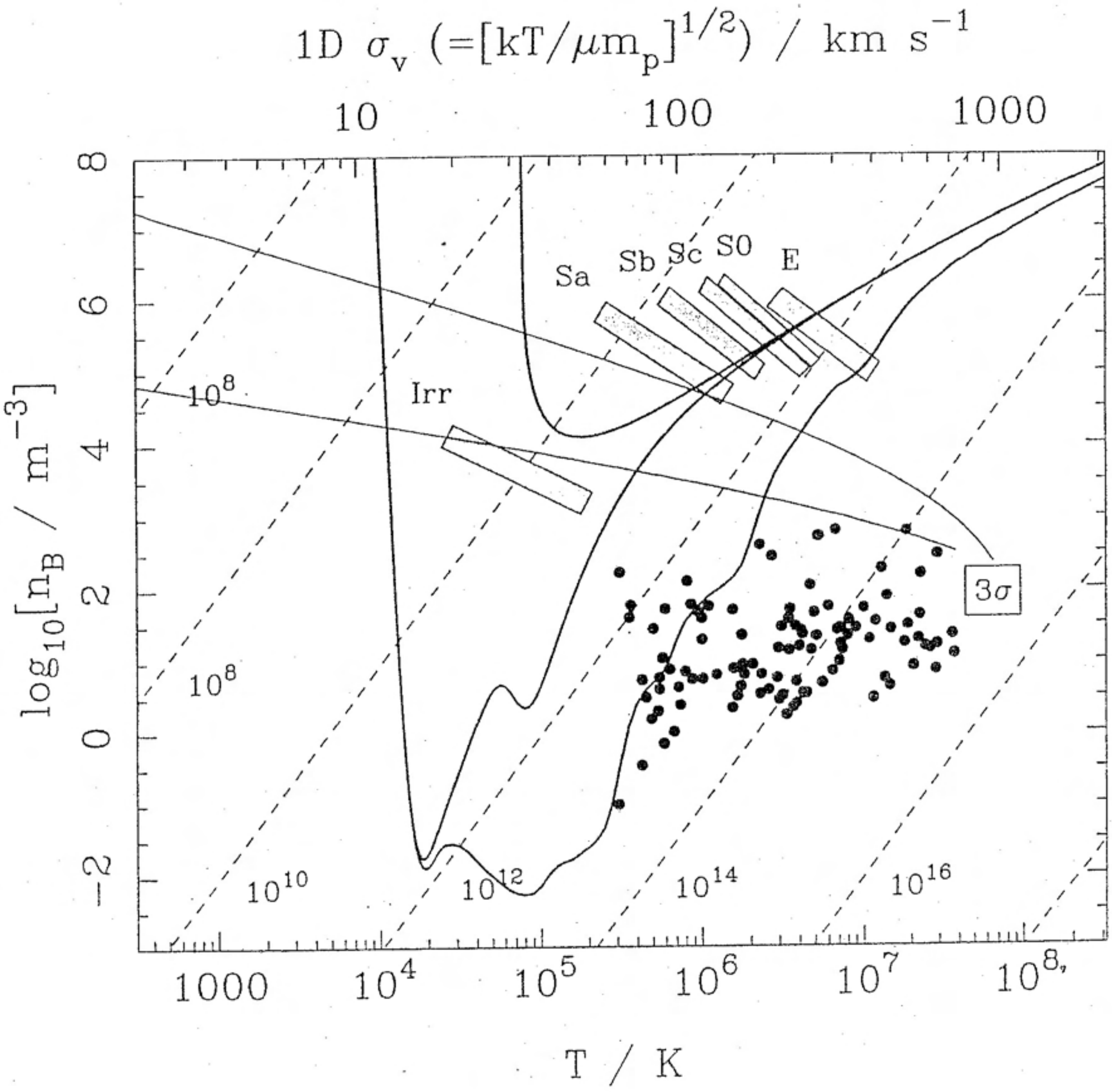
$$\tau = \frac{t_{cool}}{t_{dyn}} \sim \frac{M}{9\times10^{11}\,M_\odot}$$

$$M_{gal} \lesssim 10^{12}\,M_\odot \;\Big|\; \tau \sim 1 \to R \sim 80\,\mathrm{kpc}$$

total mass



100
10
1
$\Lambda/(10^{-23}$ erg cm^3 s^{-1})
10^4
10^5
10^6
10^7
10^8
T/K



1D σ_v (=[kT/μm$_p$]$^{1/2}$) / km s^{-1}
10
100
1000
8
6
4
2
0
−2
$\log_{10}$[n$_B$ / m^{-3}]
Sa
Sb
Sc
S0
E
Irr
10^8
10^8
10^{10}
10^{12}
10^{14}
10^{16}
3σ
1000
10^4
10^5
10^6
10^7
10^8
T / K