Astrophysical Formulae

Useful constants

Speed of light	c	$2.998\ 10^{10}$	cm sec^{-1}
Planck's constant	h	$6.626\ 10^{-27}$	erg s
Rationalized Planck's constant	$\hbar = h/2\pi$	$1.055\ 10^{-27}$	erg s
Boltzmann's constat	k	$1.381\ 10^{-16}$	erg $°\mathrm{K}^{-1}$
Electron charge	e	$4.803\ 10^{-10}$	esu
Electron rest mass	m_e	$9.110\ 10^{-28}$	g
Gravitational Constant	G	$6.673\ 10^{-8}$	dyne $\mathrm{cm}^2\ \mathrm{g}^{-2}$
Avogadro's number	N_A	$6.022\ 10^{23}$	mole^{-1}
Atomic mass unit	a.m.u.	$1.661\ 10^{-24}$	g
Gas constant	R	$8.314\ 10^{7}$	erg $°\mathrm{K}^{-1}\ \mathrm{mole}^{-1}$
Stephan-Boltzmann constant	σ	$5.670\ 10^{-5}$	erg $\mathrm{cm}^{-2}\ \mathrm{s}^{-1}\ °\mathrm{K}^{-4}$
Radiation constant	a	$7.564\ 10^{-15}$	erg $\mathrm{cm}^{-3}\ °\mathrm{K}^{-4}$
Thomson cross section	σ_T	$6.656\ 10^{-25}$	cm^2
Astronomical unit	a.u.	$1.496\ 10^{13}$	cm
Parsec	pc	$3.086\ 10^{18}$	cm
Light year	l.y.	$9.460\ 10^{17}$	cm
Solar mass	$M_\odot$	$1.989\ 10^{33}$	g
Solar radius	$R_\odot$	$6.960\ 10^{10}$	cm
Solar luminosity	$L_\odot$	$3.826\ 10^{33}$	erg s^{-1}
Electron volt	eV	$1.602\ 10^{-12}$	erg

Black body

Flux and luminosity of a black body

$$F = \sigma T^4 \qquad L = 4\pi R^2 \sigma T^4 \tag{1}$$

Magnitude system

The flux f_F (erg sec^{-1} cm^{-2}) passing through filter F with a response $A_F(\lambda)$ (fraction transmitted [i.e., 0-1] at wavelength λ), and the absolute flux f, are:

$$f_F = \int f_\lambda A_\lambda(\lambda) d\lambda \qquad f = \int f_\lambda d\lambda. \tag{2}$$

(note: with index F = filter specific, without index = bolometric).
Standard filters centered around $\lambda_U \approx 3650$Å, $\lambda_B \approx 4400$Å, $\lambda_V \approx 5480$Å, Definition of magnitudes/bolometric magnitudes:

$$m_{F,1} - m_{F,2} = -2.5 \log_{10}\left(\frac{f_{F,1}}{f_{F,2}}\right) \qquad m_1 - m_2 = -2.5 \log_{10}\left(\frac{f_1}{f_2}\right) \tag{3}$$

Normalization: $m = 0$ and $m_F = 0$ for the star Vega. Also, $m_\odot = -26.83$, $m_{Sirius} = -1.6$
Sensitivity of the eye: $m \lesssim 6$. With normalization, the magnitudes through standard filters are:

$$m_X = -2.5 \log_{10}\left[\frac{\int f_\lambda(\lambda) A_X(\lambda) d\lambda}{F_{X,\lambda 0} \int A_X(\lambda) d\lambda}\right] \quad m = -2.5 \log_{10}\left[\frac{\int f_\lambda(\lambda) d\lambda}{F_0}\right] \tag{4}$$

where $F_{X,\lambda 0} = F_{\{U,B,V,R\ or\ I\},\lambda 0} = \{4.27, 6.61, 3.64, 1.74, 0.832\} \times 10^{-9}$ erg cm^{-2} s^{-1} Å^{-1}, and $F_0 = 2.52 \times 10^{-5}$ erg cm^{-2} s^{-1} are the normalization fluxes.

Absolute magnitude: $M = m$ for $\star$ at $d = 10$ pc, i.e.,

$$M = m - 5\log_{10}\left(\frac{d}{10pc}\right). \tag{5}$$

Absolute bolometric magnitude $M_{bol} \equiv M=$ mag. of total luminosity. $M_{bol,\odot} = 4.76$.
Effective temperature T_{eff}: Temperature which gives L.
"Color": $B - V \equiv m_B - m_V=$ color index
Wien approximation for blackbody: $f \propto \exp(hc/\lambda kT)$ (at short wavelengths), giving:

$$B - V \approx \frac{2.5hc\log_{10}e}{kT_c}\left(\frac{1}{\lambda_B} - \frac{1}{\lambda_V}\right) + const \approx \frac{7090}{T_c} - 0.71 \tag{6}$$

For Vega, $B - V \equiv 0$, and $T_c \approx 10,000$°K.
Spectral Types From hot (∼50000°K) to cold (∼3000°K): OBAFGKM. Each type goes from 0 to 9. (e.g., A0..A9). Sun is a G2 star.
HR Diagram: Absolute luminosity vs. Spectral Type. Other possibility: Absolute luminosity vs. B-V. (Color magnitude diagram).

Potential Energy and Virial theorem
Gravitational potential energy of spherically symmetric mass M:

$$U_{grav} = -G\int_0^M \frac{mdm}{r} \tag{7}$$

Virial theorem:

$$\frac{1}{2}\frac{d^2I}{dt^2} = 2K + \sum_i \mathbf{F}\cdot\mathbf{r} \tag{8}$$

where I is the "spherical" moment of inertial: $I \equiv \sum_i m_i r_i^2$. K is the kinetic energy of the system: $K \equiv \sum_i m_i(d\mathbf{r}_i/dt)\cdot\mathbf{r}_i$.
Particles in gravitational field:

$$\mathbf{F}_{i,j} = -\frac{Gm_im_j}{r_{i,j}^3}(\mathbf{r}_i - \mathbf{r}_j) \tag{9}$$

And the virial theorem becomes (in steady state d/dt=0):

$$K = -\frac{1}{2}\sum_{pairs} F_{ij}\cdot(\mathbf{r}_i - \mathbf{r}_j) = \frac{1}{2}\sum_{pairs}\frac{Gm_im_j}{r_{i,j}} = \frac{1}{2}\int_0^M \frac{Gm(r)dm}{r} = -\frac{\Omega}{2} \tag{10}$$

Translational kinetic energy for nonrelativistic gas: $K = \int \frac{3}{2}PdV$.

Gas+Radiation Pressure In system with both gas and radiation pressure $P = P_g + P_r$, where $P_g = (N_0k/\mu)\rho_T$, and $P_r = \frac{1}{3}aT^4$ with $a = 4\sigma/c$. We define $\beta = P_g/P$ such that $P_r = (1-\beta)P$. The relation between pressure and density:

$$P = \left[\left(\frac{N_0k}{\mu}\right)^4 \frac{3}{a}\frac{(1-\beta)}{\beta^4}\right]^{\frac{1}{3}} \rho^{\frac{4}{3}} \tag{11}$$

Molecular Weight Molecular weight μ appearing in $P_g = (N_0k/\mu)\rho_T$ is the average weight of a particle in unit of the proton mass m_p. Given *mass* fractions n_j for specie j in the *ionized* plasma, the molecular weight is:

$$\mu = \frac{\sum_j n_jA_j}{\sum_j n_j(1+Z_j)} \tag{12}$$

each specie as an atomic mass A_j and total charge Z_j.
If one gram contains X gram of H, Y gram of He and Z gram of the rest, one has:

$$\frac{1}{\mu} \approx 2X + \frac{3}{4}Y + \underbrace{\left\langle \frac{(1+Z)}{A} \right\rangle}_{\approx 1/2} Z \tag{13}$$

<u>White Dwarfs</u>

Non-relativistic:

Degeneracy pressure:

$$P_{e,nr} = \left[\frac{1}{20} \left(\frac{3}{\pi} \right)^{2/3} \frac{h^2}{m_e m_p^{5/3} \mu_e^{5/3}} \right] \rho^{5/3} \tag{14}$$

Result using polytropes:

$$R = (1.22 \times 10^4 km) \left(\frac{\rho_c}{10^6 g\ cm^{-3}} \right)^{-1/6} \left(\frac{\mu_e}{2} \right)^{-5/6} \tag{15}$$

$$M = (0.4964 M_\odot) \left(\frac{\rho_c}{10^6 g\ cm^{-3}} \right)^{1/2} \left(\frac{\mu_e}{2} \right)^{-5/2} \tag{16}$$

or

$$M = (0.7011 M_\odot)(R/10^4 km)^{-3} (\mu_e/2)^{-5} \tag{17}$$

Relativistic:

Degeneracy pressure:

$$P_{e,r} = \left[\frac{1}{8} \frac{3}{\pi} \frac{hc}{m_p^{4/3} \mu_e^{4/3}} \right] \rho^{4/3} \tag{18}$$

Result using polytropes:

$$R = (3.347 \times 10^4 km) \left(\frac{\rho_c}{10^6 g\ cm^{-3}} \right)^{-1/3} \left(\frac{\mu_e}{2} \right)^{-2/3} \tag{19}$$

$$M = (1.457 M_\odot) \left(\frac{2}{\mu_e} \right)^2 = M_{Ch} = 3.10 \left(\frac{\hbar c}{G} \right)^{3/2} \frac{1}{m_p^2 \mu_e^2} \tag{20}$$

<u>Equations for stellar structure</u>

Integration of continuity equation (mass):

$$\frac{dm}{dr} = 4\pi r^2 \rho \tag{21}$$

Hydrostatic equation:

$$\frac{dP}{dr} = -\rho \frac{Gm(r)}{r^2} \tag{22}$$

Equation of state.

$$\text{Gas pressure}: \quad P_g = \frac{\rho k T}{\mu m_p} \qquad \text{Radiation pressure}: \quad P_r = \frac{1}{3} a T^4 \tag{23}$$

Radiation Transfer:

$$\frac{dT}{dr} = -\frac{3 k_m \rho}{16 \pi a c r^2 T^3} L \tag{24}$$

and opacity law (κ_m absorption coefficient per unit mass, i.e., cm^2/gr):

$$\text{Thomson}: \quad \kappa_m = \frac{\sigma_T}{m_p}\left(\frac{X+1}{2}\right) \qquad \text{Kramer}: \quad \kappa = \tilde{\kappa}\rho T^{-3.5} \tag{25}$$

or Convective energy transfer (i.e., adiabatic gradient):

$$\left.\frac{dT}{dr}\right|_{adiab} = -\frac{g}{c_p} = -\frac{g\mu m_p}{k}\left(\frac{\gamma-1}{\gamma}\right) \tag{26}$$

Condition for convection:

$$\frac{3}{16\pi}\frac{\kappa_m \rho L(r)}{acT^3} > \frac{Gm(r)\mu m_p}{k}\left(\frac{\gamma-1}{\gamma}\right) \tag{27}$$

Conservation of Energy:

$$\frac{dL}{dr} = 4\pi r^2 \rho \epsilon \tag{28}$$

Nuclear energy generation:

$$\epsilon \approx \tilde{\epsilon}\rho^m T^n \tag{29}$$

For pp burning $m = 1$, $n \approx 5$. For CNO burning $m = 1$, $n \approx 20$. For 3α burning $m = 2$, $n \approx 40$.

Polytropes

Polytropic approximation assumes star is described by continuity+hydrostatic+polytropic relation:

$$P = K\rho^\gamma \equiv K\rho^{(n+1)/n} \tag{30}$$

In adiabatic gas: $\gamma = c_p/c_V$, In non-relativistic mono-atomic gas $\gamma = 5/3$, $n = 1.5$. In relativistic mono-atomic gas: gas $\gamma = 4/3$, $n = 3$. Eddington standard model: $n = 3$.
Standard transformations to get Lane-Emden equation:

$$\rho = \lambda\phi^n \quad ; \quad r = \xi\ell \quad ; \quad \ell \equiv \left[\frac{(n+1)K\lambda^{(1-n)/n}}{4\pi G}\right]^{\frac{1}{2}} \tag{31}$$

and the equation itself:

$$\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{d\phi}{d\xi}\right) = -\phi^n \tag{32}$$

with boundary conditions: $\phi(0) = 1$ (thus $\lambda = \rho_c$), $d\phi/d\xi|_(\xi = 0) = 0$. Outer boundary (exists only for $n < 5$), is $\phi(\xi_1) = 0$.
Stellar radius, mass:

$$R_\star = \xi_1\ell = \left[\frac{(n+1)K}{4\pi G}\right]^{\frac{1}{2}}\lambda^{(1-n)/2n}\xi_1 \quad ; \quad M_\star = -4\pi\ell^3\lambda\xi^2\frac{d\phi}{d\xi} \tag{33}$$

Average density and central pressure::

$$\frac{\bar{\rho}}{\rho_c} = -\frac{3}{\xi_1}\frac{d\phi}{d\xi}_{\xi=\xi_1} \quad ; \quad p_c = (K\lambda^{(1-n)/n})\lambda^2 = \frac{4\pi R^2 G}{(n+1)\xi_1^2}\lambda^2 \tag{34}$$

n	ξ_1	$-\xi_1^2\left(\frac{d\phi}{d\xi}\right)_{\xi=\xi_1}$	$\frac{\rho_c}{\bar{\rho}}$
0	2.4494	4.8988	1.0000
0.5	2.7528	3.7871	1.8361
1.0	3.14159	3.14159	3.28987
1.5	3.65375	2.71406	5.99071
2.0	4.35287	2.41105	11.40254
2.5	5.35528	2.18720	23.40646
3.0	6.89685	2.01824	54.1825
3.25	8.01894	1.94980	88.153
3.5	9.53581	1.89056	152.884
4.0	14.97155	1.79723	622.408
4.5	31.83646	1.73780	6,189.47
4.9	169.47	1.7355	934,800
5.0	∞	1.73205	∞

Homologous solutions

Gas pressure dominated, homologous stars with *radiation transfer* satisfy:

$$\frac{R_2}{R_1} \approx \left(\frac{\tilde{\epsilon}_2\tilde{\kappa}_2}{\tilde{\epsilon}_1\tilde{\kappa}_1}\right)^{\frac{2}{2n+5}} \left(\frac{\tilde{\mu}_2}{\tilde{\mu}_1}\right)^{\frac{2n-15}{2n+5}} \left(\frac{M_2}{M_1}\right)^{\frac{2n-7}{2n+5}} \tag{35}$$

$$\frac{L_2}{L_1} \approx \left(\frac{\tilde{\epsilon}_2}{\tilde{\epsilon}_1}\right)^{\frac{-1}{2n+5}} \left(\frac{\tilde{\kappa}_2}{\tilde{\kappa}_1}\right)^{-\frac{2n+6}{2n+5}} \left(\frac{\tilde{\mu}_2}{\tilde{\mu}_1}\right)^{\frac{14n+45}{2n+5}} \left(\frac{M_2}{M_1}\right)^{\frac{10n+31}{2n+5}} \tag{36}$$

Nuclear Reactions Reaction rate for reaction $a + X \rightarrow Y + b + Q$ is:

$$r_{aX} = \frac{1}{(1+\delta_a X)} \frac{\rho^2 N_A^2 X_a X_X}{A_a A_X} \langle \sigma v \rangle \tag{37}$$

with X_i and A_i, the mass fractions and atomic weight of specie i, and

$$\langle \sigma v \rangle = \left(\frac{8}{\mu\pi}\right)^{1/2} \frac{1}{(kT)^{3/2}} \int_0^\infty S(E) e^{[-E/kT - b/\sqrt{E}]} dE \tag{38}$$

$S(E)$ is generally a slowly varying function of E which depends on the reaction. The Gamow Peak is:

$$E_0 = \left(\frac{bkT}{2}\right)^{2/3} = 1.2 \left(Z_a^2 Z_X^2 A_{red} T_6^2\right)^{1/3} keV \tag{39}$$

For $S(E) = S_0 = const$, one obtaines:

$$r_{ax} = \frac{n_a n_x}{A_{red} Z_a Z_x} 7 \times 10^{-19} S_0[keV\ barn] \tau^2 \exp(-\tau) s^{-1} cm^{-3}. \tag{40}$$

with A_{red} the reduced atomic weight and

$$\tau \equiv \frac{3E_0}{kT} = 42.5 \left(\frac{Z_a^2 Z_x^2 A_{red}}{T/10^{6\circ}K}\right)^{1/3} \tag{41}$$

PP Chain:

PP I

$^{1}H + {}^{1}H \rightarrow \beta^{+} + \nu + {}^{2}H$

$^{2}H + {}^{1}H \rightarrow {}^{3}He + \gamma$

$^{3}He + {}^{3}He \rightarrow 2\,{}^{1}H + {}^{4}He$

+ 13.05 MeV

(eves ~ 85%)

PP II

$^{3}He + {}^{4}He \rightarrow {}^{7}Be + \gamma$

$^{7}Be + \beta^{-} \rightarrow {}^{7}Li + \nu$

$^{7}Li + {}^{1}H \rightarrow {}^{4}H + {}^{4}He$

+ 25.7 MeV

(~15%)

PP III

$^{7}Be + {}^{1}H \rightarrow {}^{8}B + \gamma$

$^{8}B \rightarrow {}^{8}Be^{*} + \beta^{+} + \nu$

$^{8}Be^{*} \rightarrow {}^{4}He + {}^{4}He$

+19.1 MeV

(0.02%)

CNO Cycle:

$^{12}C\ (p, \gamma)\ {}^{13}N$			
$^{13}N \rightarrow {}^{13}C + \beta^{+} + \nu$			
$^{13}C\ (p, \gamma)\ {}^{14}N$			
$^{14}N\ (p, \gamma)\ {}^{15}O$	$^{14}N\ (p, \gamma)\ {}^{15}O$		
$^{15}O \rightarrow {}^{15}N + \beta^{+} + \nu$	$^{15}O \rightarrow {}^{15}N + \beta^{+} + \nu$		
$^{15}N\ (p, \alpha)\ {}^{12}C$	$^{15}N\ (p, \gamma)\ {}^{16}O$	$^{15}N\ (p, \gamma)\ {}^{16}O$	
	$^{16}O\ (p, \gamma)\ {}^{17}F$	$^{16}O\ (p, \gamma)\ {}^{17}F$	$^{16}O\ (p, \gamma)\ {}^{17}F$
	$^{17}F \rightarrow {}^{17}O + \beta^{+} + \nu$	$^{17}F \rightarrow {}^{17}O + \beta^{+} + \nu$	$^{17}F \rightarrow {}^{17}O + \beta^{+} + \nu$
	$^{17}O\ (p, \alpha)\ {}^{14}N$	$^{17}O\ (p, \gamma)\ {}^{18}F$	$^{17}O\ (p, \gamma)\ {}^{18}F$
		$^{18}F \rightarrow {}^{18}O + \beta^{+} + \nu$	$^{18}F \rightarrow {}^{18}O + \beta^{+} + \nu$
		$^{18}O\ (p, \alpha)\ {}^{15}N$	$^{18}O\ (p, \gamma)\ {}^{19}F$
			$^{19}F\ (p, \alpha)\ {}^{16}O$
CN	CNO	NO	OF

3α burning:

$^{4}He + {}^{4}He \rightarrow {}^{8}Be \quad - 22\ keV$

$^{8}Be + {}^{4}He \rightarrow {}^{12}C^{*} \quad - 282\ keV$

$^{12}C^{*} \rightarrow {}^{12}C + 2\gamma \quad + 0.66\ MeV$

Robertson-Walker Metric:

$$ds^2 = (cdt)^2 - R(t)^2\Big[\frac{dr^2}{1-Kr^2} + r^2(d\theta^2 + \sin^2\theta d\phi^2)\Big] \tag{1}$$

where $K = 0,\ \pm 1$.

Friedmann Equations:

$$\ddot{R} = -\frac{4}{3}\Big(\rho + 3\frac{p}{c^2} + \frac{\Lambda}{3}\Big)R \tag{2}$$

$$\dot{R}^2 = \frac{8}{3}\pi G\rho R^2 - Kc^2 + \frac{\Lambda}{3}R^2 \tag{3}$$

Cosmological parameters:

$$H_0 = \Big(\frac{\dot{R}}{R}\Big)_0 \tag{4}$$

$$\Omega_0 = \Big(\frac{\rho}{\rho_c}\Big)_0 \tag{5}$$

$$\rho_c = \frac{3H_0^2}{8\pi G} \tag{6}$$

$$q_0 = -\Big(\frac{\ddot{R}R}{\dot{R}^2}\Big)_0 \tag{7}$$

Sound speed:

$$v_s^2 = \Big(\frac{\partial p}{\partial \rho}\Big) \tag{8}$$

'Cosmological' equation of state:

$$p = w\rho c^2 \tag{9}$$

Hydrodynamics equations:

$$\frac{\partial \rho}{\partial t} + \nabla\cdot(\rho\vec{v}) = 0 \tag{10}$$

$$\frac{\partial \vec{v}}{\partial t} + (\vec{v}\cdot\nabla)\vec{v} = -\frac{1}{\rho}\nabla p - \nabla\phi \tag{11}$$

The joint probability distribution function of the primordial perturbation filed: $\delta(x_1), \delta(x_2), ..., \delta(x_n)$:

$$P[\delta_1, ..., \delta_n] = \frac{1}{(2\pi)^n\sqrt{\det(\mathbf{R})}}\exp[-\frac{1}{2}\vec{v}R^{-1}\vec{v}^{\dagger}] \tag{12}$$

where $\vec{\delta} = (\delta_1, \delta_2, ..., \delta_n)$ and $R_{ij} =< \delta(x_i)\delta(x_j) >= \xi(r_{ij})$.

The power spectrum and the two point correlation function:

$$\xi(\vec{r}) = \frac{1}{(2\pi)^3} \int p(\vec{k}) \exp[-\imath \vec{k} \cdot \vec{r}] \mathrm{d}^3 k \tag{13}$$

Black-body radiation:

$$e = \frac{4\sigma}{c} T^4 \tag{14}$$

$$p = \frac{1}{3} e \tag{15}$$

$$n = 0.244 \left(\frac{T}{\hbar c}\right)^3 \mathrm{cm}^{-3} \tag{16}$$

where the Stefan-Boltzmann constant is $\sigma = 5.67 \times 10^{-5} \frac{\mathrm{g}}{\mathrm{s}^3 \mathrm{deg}^4}$, e, p, n are the energy density, pressure and number density of the photons.

Red-shift:

$$1 + z = \frac{\lambda_{\mathrm{obs}}}{\lambda_{\mathrm{emitted}}} \tag{17}$$

Critical density:

$$\rho_c = \frac{3H_0^2}{8\pi G} \tag{18}$$

Fundamental constants:

$$G = 6.67 \times 10^{-8} \mathrm{cm}^3 \mathrm{g}^{-1} \mathrm{s}^{-2} \tag{19}$$

$$\hbar = 1.05 \times 10^{-27} \mathrm{erg\ s} \tag{20}$$