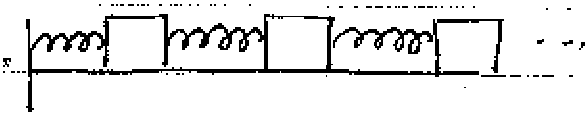
Lecture 9

The waves that we found for the strings are

Transverse Waves : the medium that carries the waves oscillate in a perpendicular direction to the direction in which the wave propagates.

We will see that EM waves are also transverse waves. However there are waves for which the directions of oscillation and wave propagation coincide. They are called Longitudinal Waves. Examples:

1. 

2. Sound Waves : as you will see in the discussion session.

Coming back to the Standing waves from last lecture you might recognize that we actually built them from two traveling waves:

$$Ae^{i\omega t}\sin kx = \frac{A}{2i}\left[e^{i(kx+\omega t)} - e^{-i(kx-\omega t)}\right]$$

left moving wave — right moving wave

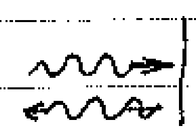
Or in "real" language $A\cos\omega t\sin kx = \frac{A}{2}\left[\sin(kx-\omega t)+\sin(kx+\omega t)\right]$

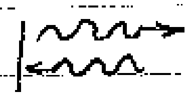
This is an example of two basic properties:

1. Superposition : ~~This is not~~ A sum of solutions to the wave equation is again a solution. This property is shared only by Linear wave equations.

2. Interference : Waves can add up to give a "super-wave" with properties quite different from the individual components. They can build up or destruct each other: like in the nodes of the standing wave.

The standing wave can also be viewed as a right moving wave which is reflected at the right boundary as left moving wave



which is then reflected by the left boundary 
and so on and so forth. We will study reflections in more details later on.

You already encountered the ~~notion~~ phenomenon of Beats in the discussion session. Let's return to it in order to introduce another fundamental notion, that of Group Velocity.

Consider the sum of 2 right moving waves:

$$A\left[\cos(k_1x-\omega_1t)+\cos(k_2x-\omega_2t)\right] = 2A\left\{\underbrace{\cos\left[\overset{k_{mod}}{\left(\frac{k_1-k_2}{2}\right)}x-\overset{\omega_{mod}}{\left(\frac{\omega_1-\omega_2}{2}\right)}t\right]}_{\text{the modulation part}}\cdot\cos\left[\left(\frac{k_1+k_2}{2}\right)x-\left(\frac{\omega_1+\omega_2}{2}\right)t\right]\right\}$$

responsible for the fast oscillations

At what frequency do the modulations propagate?

Applying similar argument to the one we used to find the phase velocity, to the modulation part we have

$$v_{mod} = \frac{\omega_{mod}}{k_{mod}} = \frac{\omega_1-\omega_2}{k_1-k_2}$$

ω is determined by the dispersion relation

$$= \frac{\omega(k_1)-\omega(k_2)}{k_1-k_2}$$

$$= \frac{d\omega}{dk} + \cdots$$

where the derivative is to be evaluated at the average wave number $\frac{k_1+k_2}{2}$

In most of the interesting applications $\omega_1-\omega_2 \ll \omega_1+\omega_2$ and thus we can usually neglects all the other terms.

$v_g = \frac{d\omega}{dk}$ is called the <u>Group Velocity</u> = velocity at which the information is transmitted

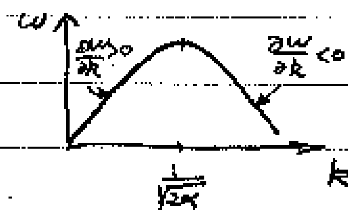
For non dispersive waves $\omega = vk$ and $v_\varphi = \frac{\omega}{k} = v$

$$\parallel$$

$$v_g = \frac{\partial \omega}{\partial k} = v$$

However, for dispersive waves $v_\varphi \neq v_g$. For example consider the wave equation

$$\frac{\partial^2 \psi}{\partial t^2} - c^2 \frac{\partial^2 \psi}{\partial x^2} - \alpha c^2 \frac{\partial^4 \psi}{\partial x^4} = 0$$



for which $\omega = kc\sqrt{1 - \alpha k^2}$

$v_\varphi = \frac{\omega}{k} = c\sqrt{1-\alpha k^2}$ depends on k : dispersive waves

$v_g = \frac{\partial \omega}{\partial k} = c\sqrt{1-\alpha k^2} - \frac{\alpha c k^2}{\sqrt{1-\alpha k^2}}$ can be positive zero or <u>even negative</u>

For such cases the modulation propagates in an opposite direction to the phase!

The phenomenon of $v_\varphi \neq v_g$ is called "Wave Dispersion" and manifests itself in the spreading of wave packets :

We can make any shape from sums over simple waves. This is just what you did when you calculated for example the Fourier transform of the function  : You expressed it as a sum over e^{ikx} with different k's. If all of these waves move with the same velocity the packet will maintain its shape. However if different components move with different velocities → dispersion it will get smeared