Lecture 7

Descrete System | Continuous System

Puzzle: We know that our system of N linear equations have N independent solutions. It seems that we obtained an infinite number of them since k is a continuous parameter. However, when we solved the equation for A_n (or $f(x)$) we didn't take into account the Boundary Conditions at $x=0$ and $x=L$. ~~When this is done~~

We will show that the boundary conditions impose specific permitted values for k - but they may be still infinite number of them. However we'll show that in the descrete case they will all be equivalent to N "elementary" solutions.

Solutions for fixed boundary condition - Standing Waves

Here we add two beads that are fixed at $x=0$ and $x=(N+1)a=L$

$$\Rightarrow \quad \psi_0(t) = \psi_{N+1}(t) = 0 \qquad\qquad \psi(0,t) = \psi(L,t) = 0$$

We can take care of the condition at $x=0$ by taking the combination

$$A_n = A\left[\frac{e^{ikna} - e^{-ikna}}{2i}\right] = A \sin nka \qquad\qquad f(x) = A\left[\frac{e^{ikx} - e^{-ikx}}{2i}\right] = A \sin kx$$

$$\boxed{\tilde{A}(k) = \frac{A}{2i}, \quad \tilde{A}(-k) = -\frac{A}{2i}}$$

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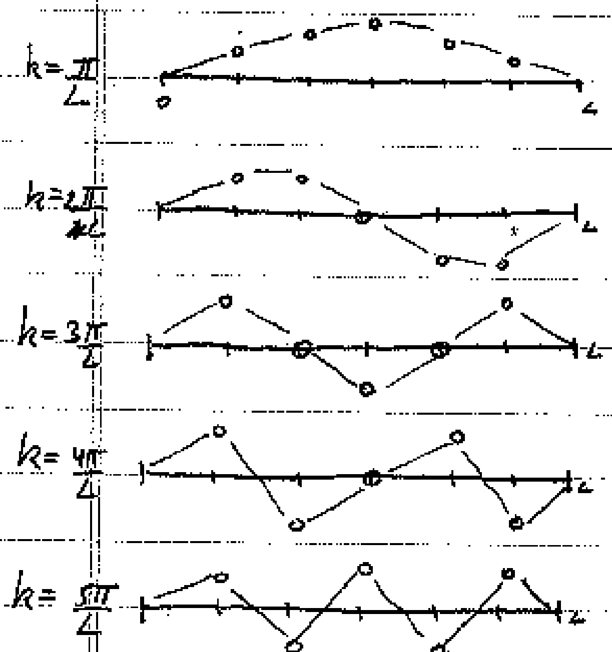
The condition at $x=L$ is satisfied if

$k = \frac{\pi}{L}, \frac{2\pi}{L}, \frac{3\pi}{L} \cdots \quad j\frac{\pi}{L}$ j an integer : k is quantized

Example $N=5$:



Since here $N \to \infty$ we have infinite allowed values of k

$j = 1,2,3 \cdots$

Note the solution with $k = 6\frac{\pi}{L}$

is the same as the one for $k=0$, the one with $k = \frac{7\pi}{L}$ is the same as $k=\frac{\pi}{L}$ and so on

We have only $N=5$ independent solutions

~~This is because~~: with $k = j\frac{\pi}{L}$ $j=1 \cdots N$

Because $\sin\left[n\frac{(N+1)\pi a}{L}\right] = \sin\left[n\pi\right] = 0$

$$\sin\left[n\frac{(N+2)\pi a}{L}\right] = \sin\left[n\pi + n\frac{\pi}{L}a\right] = (-1)^n \sin\left(n\frac{\pi}{L}a\right)$$

$\vdots$

<u>Descrete System</u> | <u>Continuous System</u>

The time dependence:

$\psi_n(t) = \left[A e^{i\omega t} + B e^{-i\omega t}\right] \sin nka$ $\qquad$ $\psi_n(x,t) = \left[A e^{i\omega t} + B e^{-i\omega t}\right] \sin kx$

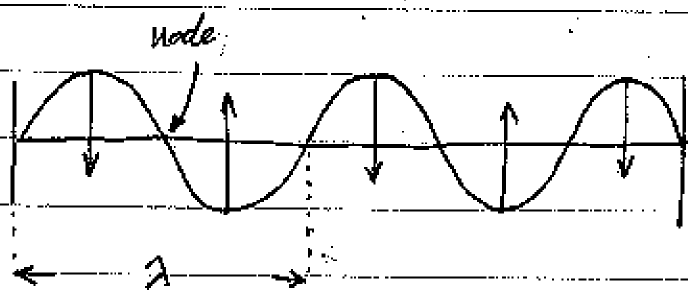
$\omega = 2\sqrt{\frac{T_0}{Ma}} \sin \frac{ka}{2}$ $\qquad$ $\omega = \sqrt{\frac{T_0}{\mu}}\, k$

$\lambda = \frac{2\pi}{k}$ is the <u>Wave Length</u> - the distance between consecutive peaks of the waves.

The continuum approximation makes sense as long as $\lambda \gg a$ $\Leftrightarrow$ $k \ll a$. In this limit we can expand $\sin \frac{ka}{2} \simeq \frac{ka}{2}$ and we get the same dispersion for the discrete system as for the continuous one.

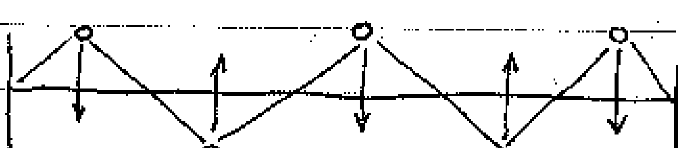
The solutions we found are <u>Standing Waves</u>. They have a fixed (sinusoidal) pattern in x, with wave length $\frac{2\pi}{k}$, with an amplitude that oscillates in time with frequency ω. (Take $A=B$ to get $\cos\omega t$ for example).

<u>Example</u>:



: <u>Cont. Sys.</u>

$k = \frac{5\pi}{L} \Rightarrow \lambda = \frac{2}{5}L$



: <u>Desc. Sys.</u> $(N=5)$