Fourier analysis using sin cos

Reading assignment: W- 2.1 2.2 (2.3) 2.4 , 4.1 4.2 } Waves in Continuous Systems

FI Chap. 47, 49 Ch 50

Differential Calculus of Vector Fields: F II Chap 2,3

# Lecture 6

So far we have been concerned with the motion of a single harmonic oscillator. In the last discussion session and in your homework you've dealt with systems consisting of two coupled oscillators. We will now generalize our treatment to systems of $N$ coupled oscillators and in particular examine the continuum limit $N \to \infty$.

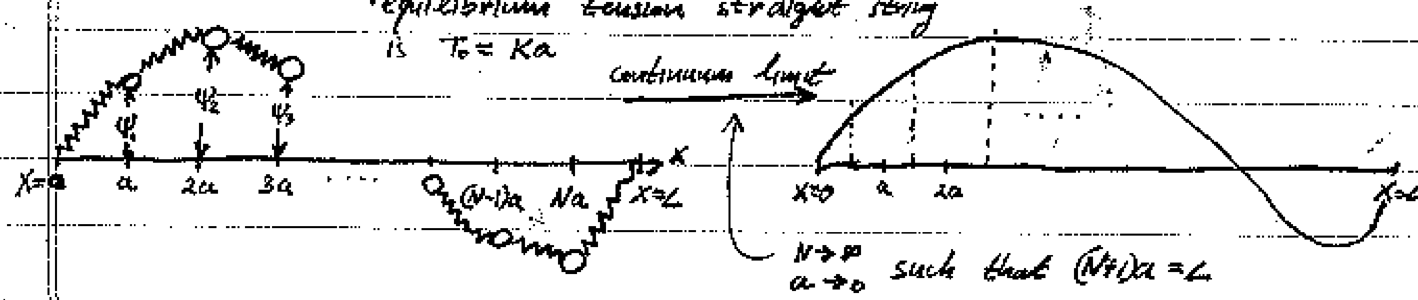
Let us consider a model of a string of length $L$ as a system of $N$ beads.

Descrete System — Continuous System

Description:

The beads are located at $x = a, 2a, \cdots Na$. The total length is $L = (N+1)a$. Each bead has mass $M$. The beads are connected by springs or string which we assume obey Hook's law (tension proportional to length).

: equilibrium tension straight string is $T_0 = Ka$



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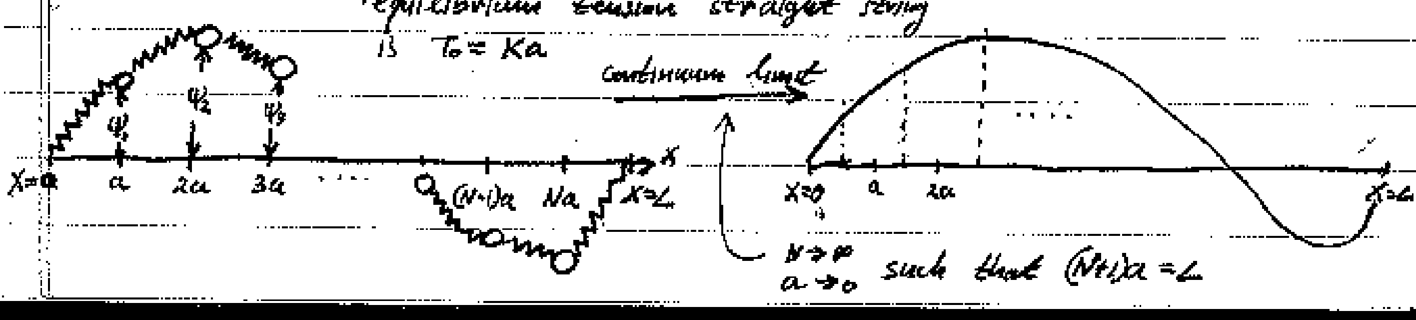


## Descrete System

## Continuous System

What are the degrees of freedom – the dependent variables that we try to find as a function of time ?

**Descrete System:** The degrees of freedom are the displacements $\psi_n(t)$ of the individual beads from their equilibrium positions.

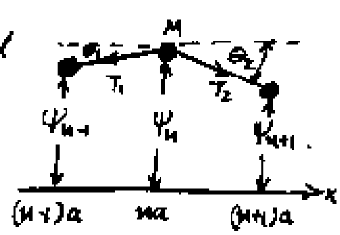
**Continuous System:** In the continuum limit the index of the beads becomes a continuous parameter: $x = na$. We are trying to find the displacements of infinitesimal elements of the string. We identify each element by its position along the x axis. $\psi_n(t) \rightarrow \psi(x,t)$

Note: x is a (continuous) index

Not a degree of freedom. We are not trying to find $x(t)$ !

### Equations of Motion

Denote by $l_{n,n+1}$ length of string between n and n+1 bead. Consider the n-th bead




In our approximation $T_1 = \frac{T_0 \cdot l_{n-1,n}}{a} = \frac{T_0}{\cos\theta_1}$

$$T_2 = \frac{T_0\, l_{n,n+1}}{a} = \frac{T_0}{\cos\theta_2}$$

$\Rightarrow F_x = T_2 \cos\theta_2 - T_1 \cos\theta_1 = 0$

↳ No motion in the x direction

**Descrete System** | **Continuous System**

$F_y = T_0 \tan\theta_2 - T_0 \tan\theta_1$

$$\Rightarrow M\frac{d^2\psi_n}{dt^2} = T_0\left[\frac{\psi_{n+1}-\psi_n}{a}\right] - T_0\left[\frac{\psi_n-\psi_{n-1}}{a}\right]$$

$$\frac{\psi_{n+1}(t)-\psi_n(t)}{a} \xrightarrow{a\to 0} \left.\frac{\partial\psi(x,t)}{\partial x}\right|_{x=na}$$

$$\frac{\psi_n(t)-\psi_{n-1}(t)}{a} \xrightarrow{a\to 0} \left.\frac{\partial\psi(x,t)}{\partial x}\right|_{x=(n-1)a}$$

$$\Rightarrow \frac{M}{a}\frac{\partial^2\psi(x,t)}{\partial t^2} = T_0\left[\frac{\left.\frac{\partial\psi}{\partial x}\right|_{x=na} - \left.\frac{\partial\psi}{\partial x}\right|_{x=(n-1)a}}{a}\right]$$

$$\xrightarrow{a\to 0} \frac{\partial^2\psi(x,t)}{\partial t^2} = \frac{T_0}{\mu}\frac{\partial^2\psi(x,t)}{\partial x^2}$$

$\mu = \frac{M}{a}$ : the linear mass density

This equation has the form of a Wave equation!
(A partial differential equation).

<u>Find the Modes</u> : Solutions in which all the beads / string elements oscillate with the same frequency.

Put $\psi_n(t) = A_n e^{i\omega t}$ | Put $\psi(x,t) = f(x)e^{i\omega t}$

$$\Rightarrow A_{n+1} + A_{n-1} = A_n\left(2 - \frac{Ma}{T_0}\omega^2\right) \qquad \frac{\partial^2 f(x)}{\partial x^2} = -\frac{\mu\omega^2}{T_0}f(x)$$

Actually $N$ coupled linear equations in the variables $A_1 \cdots A_N$ } become in the continuum limit a differential equation for $f(x)$.

boundary condition

$$\begin{bmatrix} A_0 - \left(2-\frac{Ma}{T_0}\omega^2\right)A_1 + A_2 & = 0 \\ A_1 - \left(2-\frac{Ma}{T_0}\omega^2\right)A_2 + A_3 & = 0 \\ \vdots & \vdots \\ A_{N-1} - \left(2-\frac{Ma}{T_0}\omega^2\right)A_N + A_{N+1} & = 0 \end{bmatrix}$$

boundary cond.

Descrete System | Continuous System

It seems much harder to solve the system of coupled linear equation when in the continuous case we have an equation of the form we've been solving all along!

Put $f(x) = \tilde{f}(k) e^{ikx}$ to obtain

$$\omega^2 = \frac{T_0}{\mu} k^2$$

$$\Rightarrow \omega = \pm\sqrt{\frac{T_0}{\mu}}\, k \quad : \text{A } \underline{\text{Dispersion Relation}}$$

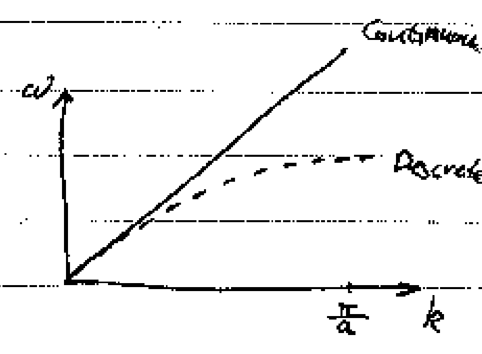
$k$ is called a <u>wave</u> and is a continuous parameter here

Knowing the solution in the continuum limit we try (noting $x = na$) $A_n \leftrightarrow f(x)$

$A_n = \tilde{A}(k) e^{ikna}$ Insert to the eq. above:

$$\Rightarrow \underbrace{e^{ika} e^{ikna} + e^{-ika} e^{ikna}}_{2\cos ka} = e^{ikna}\left(2 - \frac{Ma}{T_0}\omega^2\right)$$

$$\Rightarrow \underbrace{2(1 - \cos ka)}_{4\sin^2 \frac{ka}{2}} = \frac{Ma}{T_0}\omega^2$$




$\Rightarrow$ The dispersion relation $\omega = \pm 2\sqrt{\frac{T_0}{Ma}} \sin\frac{ka}{2}$