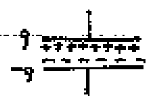
Lecture 4 (F-I ch23, W: 3.2, 3.3)

The RLC circuit

A Reminder : There are 3 basic circuit elements:

The Capacitor made, for example, out of 2 planes metallic plates seperated by small distance filled with an insulator.
Its basic property: is the linear relation between the voltage difference V between the plates and the charge $+q$ and $-q$ on them:



$$V = \frac{q}{C}$$ C: the capacitance of the capacitor

The Resistor resist the flow of electrical current like metallic wires.
Basic property: (Ohm's law) linear relation between the voltage drop V across the resistor and the current I through it:

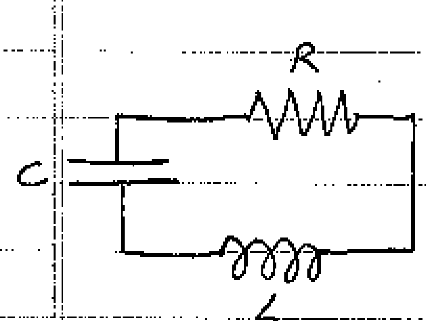
$$V = RI = R\frac{dq}{dt}$$ R: the Resistance

The Inductor : A Coil When you drive current through a coil a magnetic field develops.
A changing current causes a change in this

magnetic field and this by Faraday's induction law induces a voltage dropp across the coils ends.

Basic property $V = L\frac{dI}{dt} = L\frac{d^2q}{dt^2}$ L: the Self Inductance

Now build a closed circuit out of these 3 elements



Since there are no voltage generators in this circuit the sum of the voltage drops across the 3 elements must vanish:

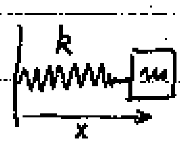
$$V_L + V_R + V_C = 0$$

$$\Rightarrow \boxed{L\frac{dq^2}{dt^2} + R\frac{dq}{dt} + \frac{q}{C} = 0}$$

Note that by writing this equation as

$$\cancel{\frac{q}{C} + R\frac{dq}{dt}} \qquad L\frac{d^2q}{dt^2} = -\frac{1}{C}\cdot q - R\frac{dq}{dt}$$

and interpreting q(t) as a coordinate X(t) of a mechanical object we can treat it as ~~se~~ Newton's second law for the mechanical object $m\frac{d^2x}{dt^2} = F = \underbrace{-kX}_{\text{spring restoring force}} - \underbrace{\gamma\frac{dx}{dt}}_{\text{friction force proportional to the velocity.}}$



If we identify :

	Mechanical Property	Electrical Property
dep. variable	$x(t)$ - position	$q(t)$ - charge
inertia	m - mass	L - inductance
resistance	γ - drag coeff.	R - resistance
stiffness	k - spring const.	$\frac{1}{C}$ - inverse capacitance

Let us now solve the equation using the complex numbers method :

Insert $q = e^{i\omega t}$ to the equation

$$\Rightarrow \quad \left[-\omega^2 + \frac{R}{L} i\omega + \frac{1}{LC}\right] e^{i\omega t} = 0$$

Have a non trivial solution if $\omega^2 - i\frac{R}{L}\omega - \frac{1}{LC} = 0$:
A quadratic equation with solutions -

$$\omega_{\pm} = \frac{-\left(-\frac{iR}{L}\right) \pm \sqrt{\left(-\frac{iR}{L}\right)^2 - 4\left(-\frac{1}{LC}\right)}}{2}$$

$$\Rightarrow \quad \omega_{\pm} = i\frac{R}{2L} \pm \sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}}$$

The general solution is then $\quad \omega' = \sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}}$

$$q(t) = A_+ e^{-\frac{R}{2L}t} e^{i\omega' t} + A_- e^{-\frac{R}{2L}t} e^{-i\omega' t}$$

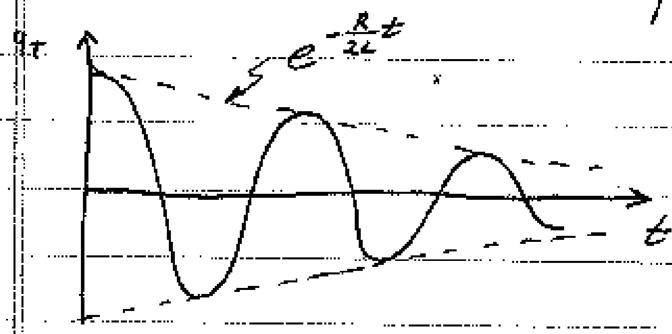
$$= e^{-\frac{R}{2L}t} \underbrace{\left[A_+ e^{i\omega' t} + A_- e^{-i\omega' t} \right]}$$

← ~~can produces an oscillatory part just as in the pendulum~~

We thus find that R causes the damping of the oscillations

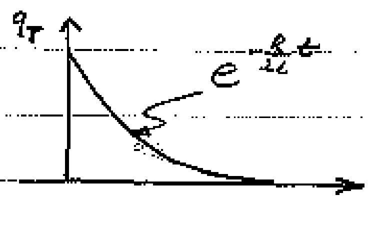
Depending on the parameters we can have different behaviours:

I. For $\frac{1}{LC} > \frac{R^2}{4L^2}$ $\Rightarrow$ ω' is real and the expression in brackets for q produces oscillatory behavior just as in the pendulum case



: damped oscillations with period ω'

II. $\frac{1}{LC} = \frac{R^2}{4L^2}$ $\Rightarrow$ $\omega' = 0$



: critically damped

III. $\frac{1}{LC} < \frac{R^2}{4L^2}$ $\Rightarrow$ ω' is imaginary $\omega' = i\omega_I$ $\omega_I = \sqrt{\frac{R^2}{4L^2} - \frac{1}{LC}}$

$$q_T = e^{-\frac{R}{2L}t}\left[A_+ e^{-\omega_I t} + A_- e^{\omega_I t}\right]$$

this term decays faster

The current decays like $e^{-\left(\frac{R}{2L} - \omega_I\right)t}$;

more slowly than the critically damped case

it's called "overdamped".