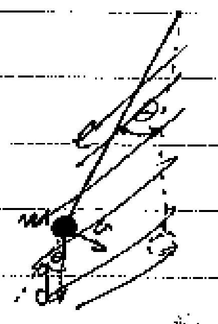
Lecture 3 (W Sec 1.2, 1.3 F-I Chap 21, 23.1)

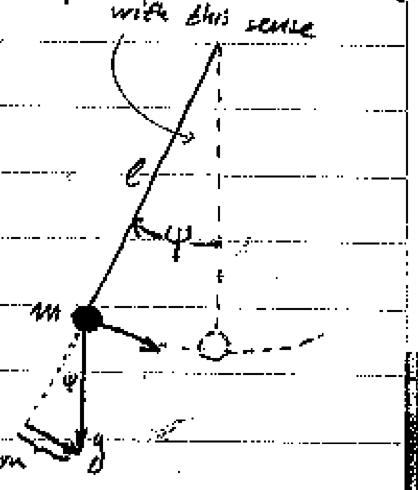
> Also read W Sec. 1.4, 1.5 in preparation for the discussion period! F-I, 48.1, 48.2, 48.3

We now want to make use of the complex numbers to study harmonic motion.
We will start by using the "conventional" technique and then move to complex variables to see the advantages.

Consider a <u>pendulum</u>

made of a massless string of length l and a mass m.





The displacement of the mass measured along the perimeter of the circular arc of its path is $l\psi$

$\Rightarrow$ The tangential velocity of the mass is $l\frac{d\psi}{dt}$
and its tangential acceleration $l\frac{d^2\psi}{dt^2}$

The return force is the tangential component of the weight (the total force along the wire vanishes) $F = -mg\sin\psi$

⇒ Newton's second law $ml\frac{d^2\psi}{dt^2} = -mg\sin\psi$

$$\Rightarrow \frac{d^2\psi}{dt^2} = -\frac{g}{\ell}\sin\psi$$

We can now use Taylor's series expansion $\sin\psi = \psi - \frac{\psi^3}{3!} + \frac{\psi^5}{5!} - \dots$
For small ψ (in radians remember) we can neglect all terms in the expansion except for the first one.
How small ψ should be? – It depends on the accuracy you want from the solution.

⇒ For <u>Small Oscillations</u> $\frac{d^2\psi}{dt^2} + \frac{g}{\ell}\psi = 0$

This is an
- <u>Ordinary</u> : contain derivatives w.r.t 1 variable only
- <u>Linear</u> : all terms contain ψ only in first power
- <u>Homogeneous</u> : no terms that don't contain ψ
- <u>Differential Equation</u>
- <u>of Order 2</u> : the highest degree of derivatives involved
- <u>with Constant Coefficients</u> : of the different terms don't depend on t.

You'll learn that an ordinary linear differential equation of order n has n independent solutions (in the sense that one can't be written as linear combinations of the others)

It is also easy to see that if ψ is a solution of the equation then $A\psi$ is also a solution with A: constant.

Thus the general solution can be written as
$\psi = A_1 \psi_1 + A_2 \psi_2 + \cdots A_n \psi_n$.

In our case $n=2$ and it is easy to check that

$\psi_1 = \cos \omega t$ and

$\psi_2 = \sin \omega t$

are two independent solutions provided $\omega^2 = \frac{g}{\ell} \Rightarrow \omega = \sqrt{\frac{g}{\ell}}$
(since $\frac{d\psi_1}{dt} = -\omega \sin \omega t$, $\frac{d^2\psi_1}{dt^2} = -\omega^2 \cos \omega t \Rightarrow$

if

$$\frac{d^2\psi_1}{dt^2} + \frac{g}{\ell}\psi_1 = \left(-\omega^2 + \frac{g}{\ell}\right)\cos\omega t = 0$$

The general solution is then $\psi = A\cos\omega t + B\sin\omega t$
and is determined by the
constants A and B which are then specified by the 2 initial conditions of the motion of the pendulum : say, its position and velocity at $t=0$.

By writing $A = a\cos\Delta$, $B = -a\sin\Delta$ we can rewrite ψ in the equivalent form

$$\psi = a\left[\cos\Delta\cos\omega t - \sin\Delta\sin\omega t\right]$$
$$= a\cos(\omega t + \Delta)$$

the amplitude of the oscillations — the frequency — the phase of the oscillation.

We are now going to apply complex numbers to our analysis by the following trick.

Let us denote by ψ_T the angle that we would measure in a true experiment, It is of course a real quantity. Mathematically we can regard it as the real part of a "fictitious" complex phase ψ : $\psi_T = \text{Re}\,\psi$

ψ doesn't really exist in the real world. It's just ~~an~~ a convinient way of doing the algebra. Its real part though is the physical angle.

For example $\quad \psi = Ae^{i\omega t} \quad \rightarrow \quad \psi_T = (\text{Re}A)\cos\omega t - (\text{Im}A)\sin\omega t$

A can be complex

We'll now see why it is usefull. Consider again the equation

$$\frac{d^2\psi}{dt^2} + \frac{g}{l}\psi = 0$$

and consider it now as an equation for the complex angle ψ. Writing $\psi = \psi_R + i\psi_i$ we have

$$\frac{d^2}{dt^2}(\psi_R + i\psi_i) + \frac{g}{l}(\psi_R + i\psi_i) = 0$$

or $$\frac{d^2\psi_R}{dt^2} + \frac{g}{l}\psi_R + i\left[\frac{d^2\psi_i}{dt^2} + \frac{g}{l}\psi_i\right] = 0$$

Now if two complex numbers are equal their real ~~and~~ ~~imaginary~~ parts should be equal and their imaginary parts should be equal.

The real and imaginary parts of 0 are 0 $\Rightarrow$

$$\frac{d^2\psi_R}{dt^2} + \frac{g}{l}\psi_R = 0$$

The real part of ψ satisfies the same equation as ψ_T.

<u>Note</u> For this to be true it is <u>essential</u> that the <u>equation is Linear</u>!

For example if there was a term like $\lambda\psi^2$ in the equation: $\frac{d^2\psi}{dt^2} + \frac{g}{l}\psi + \lambda\psi^2 = 0$ then when we substitute $\psi = \psi_R + i\psi_I$ we would get $\lambda(\psi_R + i\psi_I)^2 = \lambda(\psi_R^2 - \psi_I^2) + 2i\lambda\psi_R\psi_I$. So we see that the real part of the equation will not involve just $\lambda\psi_R^2$ now but $-\lambda\psi_I^2$ as well. In that case we get a different equation than we want to solve.

[Such terms exist in our problem. For example the first of them will be $\frac{g}{l}\psi^3$]

Let us now proceed with our linear equation and try $\psi = Ae^{i\omega t}$:

$$\frac{d^2}{dt^2}\left(Ae^{i\omega t}\right) + \frac{g}{l}Ae^{i\omega t} = 0$$

but $\frac{d}{dt}e^{i\omega t} = i\omega e^{i\omega t} \Rightarrow \frac{d^2}{dt^2}e^{i\omega t} = (i\omega)^2 e^{i\omega t} = -\omega^2 e^{i\omega t}$

$$\Rightarrow \quad -\omega^2 A e^{i\omega t} + \frac{g}{l}Ae^{i\omega t} = 0 \quad \Rightarrow$$

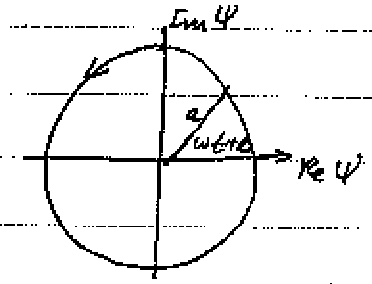
$$\left(\frac{g}{l} - \omega^2\right)Ae^{i\omega t} = 0$$

A non trivial solution ($A \neq 0$) exists if $\omega^2 = \frac{g}{l} \Rightarrow \omega = \pm\sqrt{\frac{g}{l}}$
and the general solution is

$$\psi = A_+ e^{i\sqrt{\frac{g}{l}}t} + A_- e^{-i\sqrt{\frac{g}{l}}t}$$

The coefficients A_+ and A_- are generally complex and are to be determined by the initial conditions. For example taking $A_+ = a e^{i\Delta}$, $A_- = 0$ a, Δ : real

$$\psi = a e^{i(\omega t + \Delta)} \Rightarrow \psi_r = a\cos(\omega t + \Delta)$$



* circular motion in the complex plane translates into pure oscillatory motion of the real part

The same trick can be used to solve a general ~~linear~~ linear ODE with constant coefficients:

$$k_n \frac{d^n\psi}{dt^n} + k_{n-1}\frac{d^{n-1}\psi}{dt^{n-1}} + \dots + k_1\frac{d\psi}{dt} + k_0\psi = 0$$

Take $\psi = e^{i\omega t}$ and note $\frac{d^n}{dt^n} \to (i\omega)^n$

The great thing: ODE turn into algebraic eq.

$\Rightarrow k_n(i\omega)^n + k_{n-1}(i\omega)^{n-1} + \dots k_1(i\omega) + k_0 = 0$: solve for ω: will have n roots $\omega_1 \dots \omega_n$

General solution $\psi = \sum_n A_n e^{i\omega_n t}$