## Lecture 26

The fundamental failure of statistical mechanics and Maxwell's theory to explain the spectrum of black body radiation attracted major attention. Max Planck found a function that could fit the experimental data and then tried to construct a "logical argument" that will reproduce the required function. In doing so he had to postulate a radical new concept:
The energies of the EM modes can not take arbitrary values but only descrete values given by $E = n\hbar\omega$ $n = 0, 1, 2, 3 \cdots$
where $\omega = c|\vec{k}|$ is the frequency of the mode.
$\hbar$ (h bar) is a fundamental constant of nature $\hbar = 1.06 \cdot 10^{-27}$ erg sec in CGS.
This means that we can think of an EM mode as being composed of $n$ elementary excitations each of energy $\hbar\omega$.
These elementary excitations of the EM field are called Photons and were postulated by Einstein in order to explain the photoelectric effect.

Let us calculate $\langle E \rangle$ using this new assumption. We still belibe that $P(E,T) \propto e^{-\beta E}$ but now $E = n\hbar\omega$ and $n$ could be any integer from $0$ to $\infty$.

We then have :

for each mode specified by $\vec{k}$ and $P$ we now have an integer $n_{\vec{k}p}$ that tells us what is the energy of the mode: $n_{\vec{k}p}\hbar\omega_k$ :

$$\langle E \rangle = \frac{\prod_{\vec{k}} \prod_{p} \left( \sum_{n_{\vec{k}p}=0}^{\infty} E e^{-\beta E} \right)}{\prod_{\vec{k}} \prod_{p} \left( \sum_{n_{\vec{k},p}=0}^{\infty} e^{-\beta E} \right)}$$

$\omega_k = c|\vec{k}|$

It replaces the amplitudes $A_{\vec{k},p}$ and $B_{\vec{k},p}$ we had before.

$$E = \sum_{\vec{k}} \sum_{p} n_{\vec{k}p} \hbar\omega_k$$

$$\langle E \rangle = -\frac{\partial}{\partial\beta} \ln \left[ \prod_{\vec{k}} \prod_{p} \sum_{n_{\vec{k}p}=0}^{\infty} e^{-\beta E} \right]$$

$$= -\frac{\partial}{\partial\beta} \ln \left[ \prod_{\vec{k}} \prod_{p} \sum_{n_{\vec{k},p}=0}^{\infty} e^{-\beta n_{\vec{k}p}\hbar\omega_k} \right]$$

But $\displaystyle\sum_{n=0}^{\infty} e^{-\beta n\hbar\omega} = \sum_{n=0}^{\infty} \left( e^{-\beta\hbar\omega} \right)^n = \frac{1}{1-e^{-\beta\hbar\omega}}$

where we used $\displaystyle\sum_{n=0}^{\infty} X^n = \frac{1}{1-X}$ for $|X|<1$.

$$\Rightarrow \langle E \rangle = -\frac{\partial}{\partial\beta} \ln \left[ \prod_{\vec{k}} \prod_{p} \frac{1}{1-e^{-\beta\hbar\omega_k}} \right]$$

$$= \sum_{\vec{k}} \sum_{p} \hbar\omega_k \frac{e^{-\beta\hbar\omega_k}}{1-e^{-\beta\hbar\omega_k}}$$

Thus in contrast to the classical result $\frac{1}{\beta} = k_B T$ for the average energy of a mode the quantum result is:

$$\hbar\omega_k \frac{e^{-\frac{\hbar\omega_k}{k_B T}}}{1 - e^{-\frac{\hbar\omega_k}{k_B T}}}$$

it now depends on the frequency of the mode!

Let's see what happens when the fundamental energy packet $\hbar\omega_k$ of the mode is much smaller than $k_B T$. Then

for $\hbar\omega_k \ll k_B T$: $e^{-\frac{\hbar\omega_k}{k_B T}} \simeq 1$ , $1 - e^{-\frac{\hbar\omega_k}{k_B T}} \simeq \frac{\hbar\omega_k}{k_B T}$

and the average energy of the mode is $\simeq \hbar\omega_k \cdot \frac{k_B T}{\hbar\omega_k} = k_B T$ : the classical result!

But for $\hbar\omega_k \gg k_B T$ things are very different. In this limit the average energy of the mode vanishes according to $\hbar\omega_k e^{-\frac{\hbar\omega_k}{k_B T}} \ll k_B T$ : Here the classical result fails completely.

Using the same manipulations as in the classical case we find

$$\frac{d\langle E\rangle/V}{d\omega} = \frac{1}{\pi^2}\frac{\omega^2}{c^3} \cdot \hbar\omega \frac{e^{-\frac{\hbar\omega}{k_B T}}}{1 - e^{-\frac{\hbar\omega}{k_B T}}} = \frac{\hbar}{\pi^2 c^3}\frac{\omega^3}{e^{\frac{\hbar\omega}{k_B T}} - 1}$$

which is the function that is observed experimentally.

The total averge energy is now

$$\langle E\rangle = V\cdot\frac{\hbar}{\pi^2 c^3}\int_0^\infty d\omega\,\frac{\omega^3}{e^{\frac{\hbar\omega}{k_BT}}-1}$$

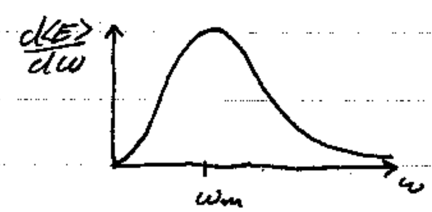
Defining $x=\frac{\hbar\omega}{k_BT}$ we get

$$= V\,\frac{1}{\pi^2\hbar^3 c^3}(k_BT)^4\underbrace{\int_0^\infty dx\,\frac{x^3}{e^x-1}}_{\text{a pure, finite number}}$$

$\Rightarrow$ The energy of the EM field in contact with a body at temperature $T$ is finite and more importantly is proportional to $T^4$ - Stephen Boltzman law

This result is important in determining how objects cool down by radiating EM waves. Hotter bodies radiate much more energy, and cool down faster.

The frequency at which they emit the maximum energy is also temperature dependent




Let's see what it is:

We wand $\frac{d}{d\omega}\left[\frac{d\langle E\rangle}{d\omega}\right]=0$

$$\Rightarrow \frac{3\omega^2}{e^{\frac{\hbar\omega}{k_BT}}-1}-\frac{\hbar\omega^3}{k_BT}\frac{e^{\frac{\hbar\omega}{k_BT}}}{\left(e^{\frac{\hbar\omega}{k_BT}}-1\right)^2}=0 \Rightarrow 1-e^{-\frac{\hbar\omega}{k_BT}}=\frac{1}{3}\frac{\hbar\omega}{k_BT}$$

$\Rightarrow \omega_m = \underbrace{\#}_{\text{some number}}\cdot\frac{k_BT}{\hbar}$ : hotter bodies radiate as higher frequencies.