## Lecture 24

When a particle is moving under the influence of an external force its energy changes due to the work done by the force:

$$\underbrace{\Delta W}_{\text{work done by force}} = \int_{x_i}^{x_f} dx\, F = \int_{x_i}^{x_f} dx \frac{d}{dt}(mv) = \underbrace{\Delta E}_{\text{change in energy of the particle}}$$

remember: $m = m(v)$

Here we consider the case where the motion takes place along the x direction only ($F = F_x$).

But:

$$\Delta E = \int_{x_i}^{x_f} dx \frac{d}{dt}(mv) = \int_{t_i}^{t_f} dt \frac{dx}{dt} \frac{d}{dt}(mv)$$

and we have $\frac{dx}{dt} = v$ and $\frac{d}{dt}(mv)\,dt = d(mv)$ so

$$\Delta E = \int_{(mv)_i}^{(mv)_f} d(mv) \cdot v$$

However $d(mv) = v\,dm + m\,dv$ and we know that $m = \frac{m_0}{\sqrt{1-\left(\frac{v}{c}\right)^2}}$

$$\Rightarrow dm = \frac{m_0 \frac{v}{c^2}}{\left[1-\left(\frac{v}{c}\right)^2\right]^{3/2}} dv = \frac{m \frac{v}{c^2}}{1-\left(\frac{v}{c}\right)^2} dv \Rightarrow dv = \frac{1-\left(\frac{v}{c}\right)^2}{m \frac{v}{c^2}} dm$$

$$\Rightarrow \quad d(m\upsilon) = \left[\upsilon + \frac{1-\left(\frac{\upsilon}{c}\right)^2}{\frac{\upsilon}{c^2}}\right] dm = \frac{c^2}{\upsilon}\, dm$$

and this leads to the result

$$\Delta E = \int_{m_i}^{m_f} \frac{c^2}{\upsilon}\, dm \cdot \upsilon = \int_{m_i}^{m_f} dm \cdot c^2 = c^2(m_f - m_i) = \Delta m \cdot c^2$$

a true differential.

$\Rightarrow$ The (change in the) energy of the particle equals (the change in) its mass times $c^2$.

$$E = mc^2$$

This formula tells us that the particle has an ~~energy~~ intrinsic energy of $m_0 c^2$ in its rest frame. This is its <u>Rest Energy</u> it's coming entirely from its rest mass.

The <u>Kinetic Energy</u> is the extra energy above the rest energy i.e.

$$KE = (m - m_0)\, c^2$$

$$= m_0 \left[\frac{1}{\sqrt{1-\left(\frac{\upsilon}{c}\right)^2}} - 1\right] c^2$$

What happens at low velocities of the types we encounter in everyday life $\left(\frac{\upsilon}{c}\right)^2 \ll 1$ ? We can then expand to have:

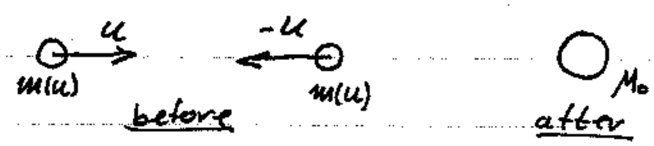
$$KE \simeq m_0\left[1+\frac{1}{2}\left(\frac{v}{c}\right)^2-1\right]c^2 = \frac{1}{2}m_0 v^2 \quad :$$

our familiar form obtained from Newtonian dynamics.

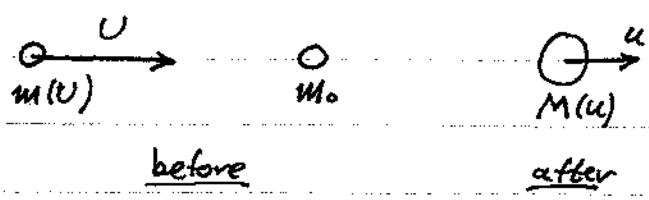
Let us use the expressions we derived for the energy and momentum to check that they are conserved in an inelastic collision.

Consider a completely inelastic collision between 2 identical particles of rest mass $m_0$. There will be a frame $S'$ in which the particles approach each other along a straight line with equal and opposite velocities of magnitude $u$.
There will then exist another frame moving relative to $S'$ with velocity $-u$ in which one of the particles is initially at rest:

In $S'$:




In $S$:




Conservation of momentum tells us that in $S'$ that the composite particle formed at the collision will be at rest.
Conservation of energy then gives

$$M_0 = 2m(u) = 2\frac{m_0}{\sqrt{1-\left(\frac{u}{c}\right)^2}}$$

Let us see what happens in $S$:

Using the relativistic law of transformation of velocities we find that in $S$ the velocity of the first particle is

$$U = \frac{u+u}{1+\frac{u\cdot u}{c^2}} = \frac{2u}{1+\frac{u^2}{c^2}} = \frac{2uc^2}{c^2+u^2}$$

and that the velocity of the composite particle is $u$.
Check conservation of ~~momentum~~:

initial momentum: $$m(U)\cdot U = \frac{m_0}{\sqrt{1-\left(\frac{U}{c}\right)^2}}\, U = \frac{m_0 c}{\sqrt{c^2-\left(\frac{2uc^2}{c^2+u^2}\right)^2}}\,\frac{\cdot 2uc^2}{c^2+u^2}$$

$$= \frac{2m_0c^3u}{\sqrt{c^2(c^2+u^2)^2-4u^2c^4}} = \frac{2m_0uc^2}{c^2-u^2} = \frac{2m_0}{1-\left(\frac{u}{c}\right)^2}\cdot u$$

which indeed equals the final momentum $M(u)\cdot u = 2\frac{m_0}{\sqrt{1-\left(\frac{u}{c}\right)^2}}\cdot\frac{1}{\sqrt{1-\left(\frac{u}{c}\right)^2}}\cdot u$

What about energy conservation?

initial energy: $$m(U)\cdot c^2 + m_0c^2 = m_0c^2\left(1+\frac{1}{\sqrt{1-\left(\frac{U}{c}\right)^2}}\right)$$

$$= m_0c^2\left[1+\frac{(c^2+u^2)}{c^2-u^2}\right] = \frac{2m_0c^4}{c^2-u^2} = \frac{2m_0}{1-\left(\frac{u}{c}\right)^2}\cdot c^2$$

which indeed equals the final energy $M(u)c^2 = \frac{2m_0}{\sqrt{1-\left(\frac{u}{c}\right)^2}}\cdot\frac{1}{\sqrt{1-\left(\frac{u}{c}\right)^2}}\cdot c^2$

Can we find a relation between the relativistic energy and momentum? Let's try:

$$P = mv = \frac{m_0 v}{\sqrt{1-\left(\frac{v}{c}\right)^2}} \quad \Rightarrow \quad P^2\left[1-\left(\frac{v}{c}\right)^2\right] = m_0^2 v^2$$

$$\Rightarrow \quad \frac{P^2}{c^2}\left[1-\left(\frac{v}{c}\right)^2\right] = -m_0^2\left[1-\left(\frac{v}{c}\right)^2\right] + m_0^2 \quad \Rightarrow \quad 1-\left(\frac{v}{c}\right)^2 = \frac{m_0^2}{m_0^2 + \frac{P^2}{c^2}}$$

$$E^2 = m^2c^4 = \frac{m_0^2 c^4}{1-\left(\frac{v}{c}\right)^2} = c^4\left(m_0^2 + \frac{P^2}{c^2}\right) = m_0^2c^4 + P^2c^2$$

$$\Rightarrow \quad E = \sqrt{m_0^2c^4 + P^2c^2}$$

This is an important relations that indicates that it is possible to have "particles" with zero mass ($m_0 = 0$) and yet for such objects to have perfectly defined energy and momentum

For $m_0 = 0$ : $\quad E = Pc \quad$ or $\quad P = \frac{E}{c}$

This is exactly the case for the photons : the "Particles" of the EM field. For them we find

$$E = \hbar\omega \qquad \hbar = \frac{h}{2\pi} \leftarrow \text{Planck's constant}$$

$\Rightarrow \quad P = \frac{\hbar\omega}{c} \quad$ but we know that the dispersion law of light is $\omega = ck$

$\Rightarrow \quad P = \hbar k \quad$ is the momentum associated with a photon.