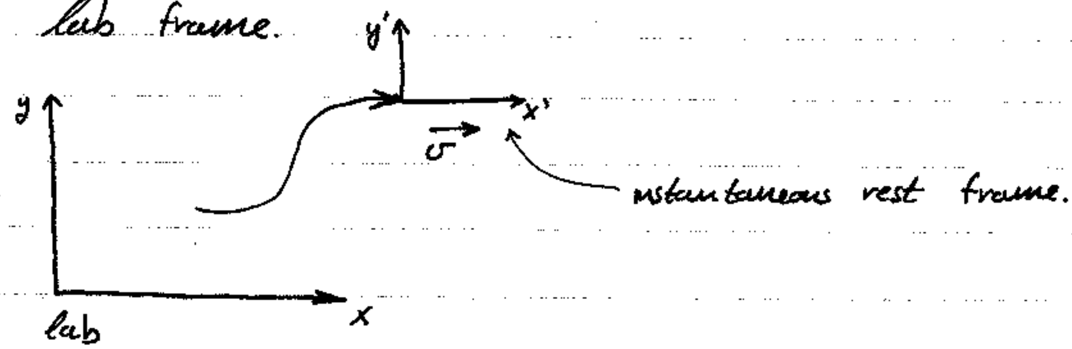
Lecture 23

So far we have considered only relativistic kinematics.
We would like to investigate now the dynamics in special relativity.

Consider a particle moving under the influence of a force.
Its velocity changes in time but at each instant we can consider its motion in its instantaneous rest frame ie. the frame that is moving with the same velocity of the particle at that moment.

Assume that at the moment we are considering the particle is moving with velocity v along the x direction as measured in the lab frame.



In the instantaneous rest frame the velocity of the particle is zero.
We thus expect Newtonian dynamics to be correct in this local frame. In particular lets consider Newton's second law of motion.
We want to write it in a way that imply the law of conservation of momentum:

The x component reads:

$$\frac{d}{dt'} P_{x'} = F_{x'} \quad \Rightarrow \text{ No force } P \text{ is conserved}$$

But in Newtonian dynamics the momentum is given by the mass of the particle times its velocity

$$\Rightarrow P_{x'} = m_0 \frac{dx'}{dt'}$$

m_0 is called the Rest Mass of the particle: It's the mass measured in its rest frame

$$\Rightarrow m_0 \frac{d^2x'}{dt'^2} = F_{x'}(x', t')$$

Let us transform this equation to the lab frame:

$$\frac{d}{dt'} = \frac{\partial x}{\partial t'}\frac{\partial}{\partial x} + \frac{\partial t}{\partial t'}\frac{\partial}{\partial t} = \gamma v \frac{\partial}{\partial x} + \gamma \frac{\partial}{\partial t}$$

$$\frac{dx'}{dt'} = \left(\gamma v \frac{\partial}{\partial x} + \gamma \frac{\partial}{\partial t}\right) \gamma (x - vt) = \gamma^2 v + \gamma^2 \frac{dx}{dt} - \gamma^2 v = \gamma^2 \frac{dx}{dt}$$

$$\frac{d^2x'}{dt^2} = \left(\gamma v \frac{\partial}{\partial x} + \gamma \frac{\partial}{\partial t}\right) \gamma^2 \frac{dx}{dt} = \gamma^3 \frac{d^2x}{dt^2}$$

γ is a constant depending on the velocity of the IRF - doesn't change in time the position x' of the particle in this frame does.

$$\Rightarrow m_0 \gamma^3 \frac{d^2x}{dt^2} = F_x(x,t)$$

This is the form of the force after we transform it to the lab frame.

We want to write this equation in a way that suggests momentum conservation.

To this end note: $\frac{d}{dt}\left[\frac{m_0 v}{\sqrt{1-\left(\frac{v}{c}\right)^2}}\right] = \frac{m_0}{\sqrt{1-\left(\frac{v}{c}\right)^2}}\frac{dv}{dt} + \frac{m_0\left(\frac{v}{c}\right)^2}{\left(1-\left(\frac{v}{c}\right)^2\right)^{3/2}}\frac{dv}{dt}$

$$= \frac{m_0}{\sqrt{1-\left(\frac{v}{c}\right)^2}}\left[1 + \frac{\left(\frac{v}{c}\right)^2}{1-\left(\frac{v}{c}\right)^2}\right]\frac{dv}{dt}$$

$$= \frac{m_0}{\left(1-\left(\frac{v}{c}\right)^2\right)^{3/2}}\frac{dv}{dt} = m_0\gamma^3\frac{dv}{dt}$$

Thus we have $\frac{d}{dt}\left[\frac{m_0}{\sqrt{1-\left(\frac{v}{c}\right)^2}}v\right] = F_x$

which we can interpret since $v = v_x$ as $\frac{d}{dt}P_x = F_x$

with

$$P_x = m v_x \qquad m = \frac{m_0}{\sqrt{1-\left(\frac{v}{c}\right)^2}} = \gamma m_0$$

Generalizing this result to arbitrary direction we will find the relativistic expressions for the momentum and mass

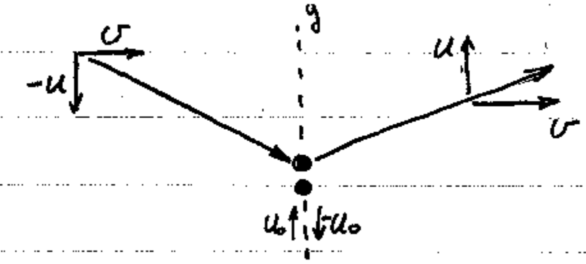
$$\boxed{\vec{P} = m(v)\vec{v} \qquad m(v) = \frac{m_0}{\sqrt{1-\left(\frac{v}{c}\right)^2}}}$$

Note that the mass of the particle as measured in the lab frame $m \to \infty$ as $v \to c$.

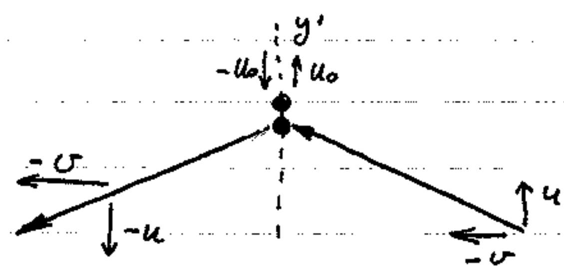
Let us check this expression for the momentum by analyzing an elastic collision between two identical particles of rest mass m_0.

Imagine an observer in S sending its particle with velocity u_0 along the y axis of its frame.
An observer in S' which is moving with velocity v along the x direction is launching its particle along its negative y' axis with velocity u_0. The two particles collide elastically.
The collision

observed in S:



observed in S':



Conservation of momentum in S (the y component) requires

$$m(u_0)\,u_0 - m(V)\,u = -m(u_0)\,u_0 + m(V)\,u \qquad \text{where } V = \sqrt{u^2+v^2}$$

$$\Rightarrow \quad \frac{m(V)}{m(u_0)} = \frac{u_0}{u}$$ Is this correct? Let's see:

$$\frac{m(V)}{m(u_0)} = \frac{\sqrt{1-\left(\frac{u_0}{c}\right)^2}}{\sqrt{1-\frac{u^2+v^2}{c^2}}} \overset{\uparrow}{=} \sqrt{\frac{c^2-u_0^2}{c^2-v^2-\left(1-\frac{v^2}{c^2}\right)u_0^2}} = \sqrt{\frac{c^2-u_0^2}{(c^2-u_0^2)\left(1-\frac{v^2}{c^2}\right)}} = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}} = \gamma \ \checkmark$$

here we use our knowledge about velocity transformations

$$u_y = \frac{u_y'/\gamma}{1+v u_x'/c^2} \qquad u_y = u \quad u_y' = u_0 \quad u_x' = 0 \quad \Rightarrow \quad u = u_0/\gamma$$

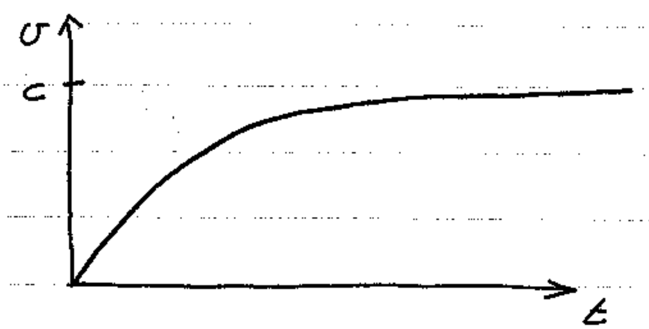
Let us now see what happens to a particle under the action of a constant force F (say everything is along the x direction)

We then have $\frac{dp}{dt} = F$

$$\Rightarrow \qquad \frac{d}{dt} \frac{m_0 v}{\sqrt{1-\left(\frac{v}{c}\right)^2}} = F$$

$$\Rightarrow \frac{m_0 v}{\sqrt{1-\left(\frac{v}{c}\right)^2}} = Ft \quad \Rightarrow \frac{\left(\frac{v}{c}\right)^2}{1-\left(\frac{v}{c}\right)^2} = \left(\frac{Ft}{m_0 c}\right)^2$$

$$\Rightarrow \left(\frac{v}{c}\right)^2 = \frac{\left(\frac{Ft}{m_0 c}\right)^2}{1+\left(\frac{Ft}{m_0 c}\right)^2} \xrightarrow{\text{for } t\to\infty} 1$$



We can't push a massive particle to the velocity of light no matter how strong the force is. The velocity will approach c asymptotically but will never reach it.
The mass increases with v it's becoming harder and harder to further accelerate the particle.