Lecture 21

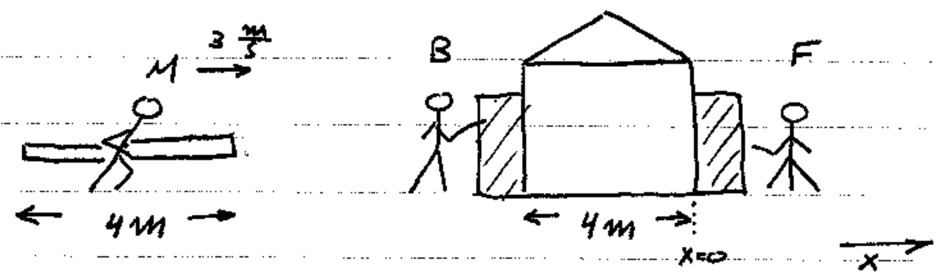
Most of us find it difficult to adjust our mind to the relativistic way of thinking. This is not surprising considering the fact that our physical intuition is build upon our daily life experience that involves velocities much smaller than c.
The only solution (like everything else in life) is to practice so we will do just that.

Consider the following example (for this example we'll assume that the speed of light is $c = 5 \frac{m}{sec}$).

Three people own some boards 5 m long and a barn 4 m wide. They know that moving objects contract in length. Hence, they decide to move the boards at a speed $v = 3 \frac{m}{sec}$ so that the length of the boards will be contracted to

$$l = l_0 \sqrt{1-\frac{v^2}{c^2}} = 5\sqrt{1-\left(\frac{3}{5}\right)^2} = 5 \cdot 0.8 = 4\,m$$

Thus, they think, the boards will fit into the barn. They station one person (B) at the back door of the barn and another (F) at the front door. The third (M) will run with the board. F and B are to close the doors simultaneously when the board is exactly inside the barn - They can do so because as we learned measuring the length of the board (which is essentially what they are doing) means reading the coordinates of its ends at the same time given by sinchronized clocks in B and F's frame.



The barn and the board as measured by observers who are at rest relative to the barn: frame S

The apparent paradox is that $\underbrace{\text{M's}}_{}$ observers in frame moving with the board, see the board a 5 m long and the barn as having a contracted length of only $4 \cdot 0.8 = 3.2$ m. How can the two set of observations be compatible?

Lets start by considering the situation as measured by B and F. We'll denote by $x=0$ the x coordinate of the front door (F) in frame S and by $t=0$ the time of the event when the front end of the board reaches F (again by S time).

F and B observe:

1. The event of the front end of the board reaching F has coordinates $(x=0, t=0)$ and the event of the back end of the board reaching B has coordinates $(x=-4, t=0)$.
$\Rightarrow$ The two events are simultaneous in their frame and they conclude that the barn and the board has the same length.
2. The observe the ~~door~~ board crashing into the closed ~~back~~ front door just after both doors are closed.

However observers in M frame (call it S') measure that

1. The barn is 3.2 m long and the board is 5 m long.
2. Assuming that the origins of S and S' coincide when the front end of the board reaches F , M will assign the coordinates $(x'=0, t'=0)$ to this event.

But what about the event of the back end of the board reaching B ?

Let us use Lorentz transformation law to find the coordinates of this event in S':

$$x' = \frac{1}{0.8}(-4) = -5 \text{ m} \qquad \gamma = \frac{1}{0.8}$$

$$t' = \frac{1}{0.8}\left(-\frac{3(-4)}{5^2}\right) = 0.6 \text{ sec}$$

as it should be : this is just the coordinate of the back end of the board in its rest frame.

$\Rightarrow$ The events of the doors being closed are <u>NOT</u> simultaneous in S'. What happens according to M is that F indeed shut the door as the front end of the board reaches it but then the board smashes through it and only 0.6 sec later when the back end reaches B the other door is closed.

Another way of arriving at the same conclusion is to realize that in S', B is at coordinates $(X' = -3.2,\ t' = 0)$ when the front end reaches F and the back of the board is still $5 - 3.2 = 1.8\,m$ outside the barn

⇛ Remember this is the meaning of measuring the length of something in special relativity: You have to do it in the same time at different places at your reference frame. According to M the barn is moving towards him at $3\ \frac{m}{sec}$. Now in the extra 0.6 sec until the second event the barn will move $0.6 \cdot 3 = 1.8\,m$. This is just the distance that places the back of the board inside the barn when B shuts the door.

All the observers agree that the back door was shut just before the board reached it and that the front door was shut after the end of the board was inside the barn.

The observers disagree on when the board crashed through the back door. F and B say it happened just after they shut the doors simultaneously, while M says it happend just after F shut the door but before B shut his. : This disagreement is real and is typical of those that can arise when two sets of observers view events that occur in different positions

Relativistic Transformation of Velocities

Once we have the Lorentz transformation for distance and time it is straightforward velocities = time derivatives of displacements as measured in two different inertial frames

We begin with the following basic equations:

$$x = \gamma(x' + v t')$$

(*) $$y = y'$$

$$t = \gamma\left(t' + \frac{v}{c^2}x'\right) \quad ; \quad \gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}};$$

Suppose that an object has velocity components u_x', u_y' as measured in S'. By the definition of velocity we have

$$u_x' = \frac{dx'}{dt'} \qquad u_y' = \frac{dy'}{dt'}$$

What will be the velocity of the object as measured in the frame S which has itself velocity $-v$ relative to S' along the x axis ?

Differentiating (*) with respect to t' we get

$$dx = \gamma\left(\frac{dx'}{dt'} + v\right)dt' = \gamma(u_x' + v)\,dt'$$

$$dy = \frac{dy'}{dt'}\,dt' = u_y'\,dt'$$

$$dt = \gamma\left(1 + \frac{v}{c^2}\frac{dx'}{dt'}\right)dt' = \gamma\left(1 + \frac{v}{c^2}u_x'\right)dt'$$

$$\Rightarrow \quad u_x = \frac{dx}{dt} = \frac{u_x' + v}{1 + \frac{v u_x'}{c^2}} \quad ; \quad u_x' = \frac{u_x - v}{1 - \frac{v u_x}{c^2}}$$

$$u_y = \frac{dy}{dt} = \frac{u_y'/\gamma}{1 + \frac{v u_x'}{c^2}} \qquad u_y' = \frac{u_y/\gamma}{1 - \frac{v u_x}{c^2}}$$

It's obvious that the expression for u_z is similar to the one for u_y with $y \to z$.

This is the relativistic law for addition of two velocities.
Let us consider the equation for u_x first. This is the law for the addition of two velocities that are in the same direction.
If both u_x' and v are small compared to c the term $\frac{v u_x'}{c^2}$ is of second order in "smallness" and most of the times can be dropped, giving us back $u_x = u_x' + v$: the Galilean result.
Consider, however, the other extreme where $u_x' = c$ i.e. a light ray measured in S'. What will be the velocity of the light measured in S?

$$u_x = \frac{c + v}{1 + \frac{v \cdot c}{c^2}} = c\,\frac{c + v}{c + v} = c$$

The same velocity c! $\Rightarrow$ The velocity of light is independent of the motion of an inertial observer.