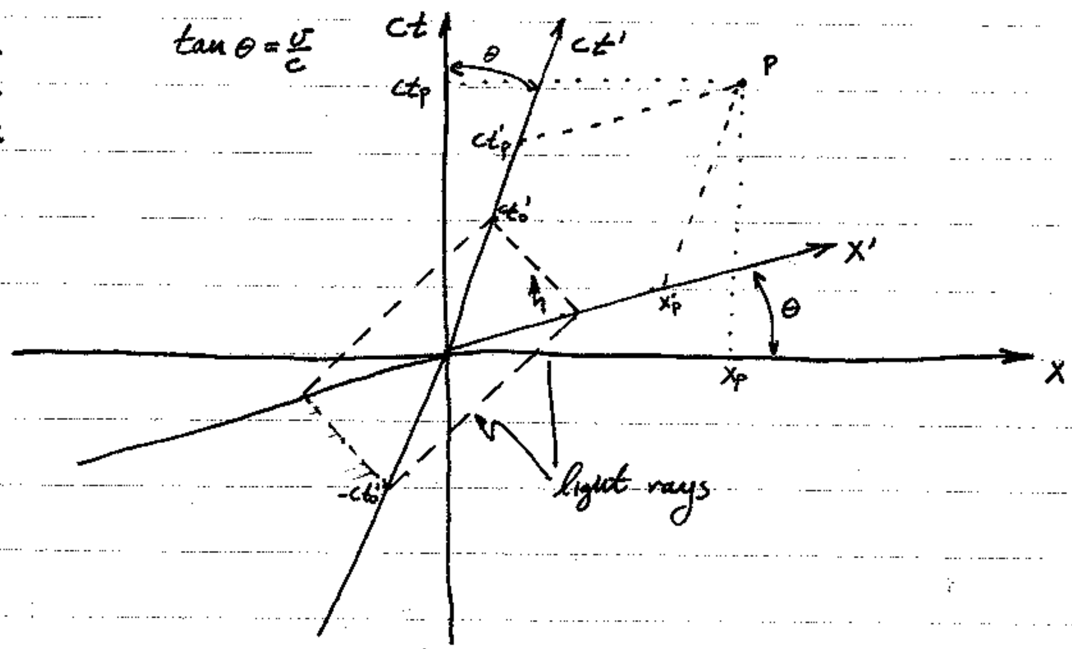
Lecture 20

Let us construct the t' and x' axes of an inertial frame S' moving with velocity v in the positive x direction relative to S. The t' axis is easy. It is just the line describing the events of the origin of S': $(x'=0)$. It is given by the line $x = vt$ (if we assume, as we will, that the origins of S and S' coincide at $t=0$).

A Minkowski diagram:



What about the x' axis? This is the line that connects all the points corresponding to $t'=0$. From Einstein's prescription for synchronizing clocks we know that these points are the ones at which light that was emitted from the origin of S at time $-t'_0$ is reflected and reaches the origin again at time t'_0. From our definition we assign to the event of the light reflection the time $t' = \left(\frac{-t'_0 + t'_0}{2}\right) = 0$. I indicated 2 such events on the diagram.

I plotted the ct axis rather than the t axis in order for the two axes (ct and x) to have the same dimensions (length). By doing so light rays are described by lines making 45° angle with the x and ct axes. You can see that the ct' axis is making an angle θ, with $\tan\theta = \frac{v}{c}$, with the ct axis. Similarly, the x' axis is making an angle θ with the x axis.

From the figure we can immediately detect that simultaneous events in S, i.e. events with the same t' coordinate (like the events defining the x' axis) are not simultaneous in S: they don't have the same t coordinate. <u>Simultaneity is Relative.</u>

Our construction implies that x' and t' should be linear functions of x and t and vice versa.
The symmetry implied by the relativity principle means that the form of the transformation should be as follows

$$(*) \quad x = ax' + bt' \quad ; \quad x' = ax - bt$$

These are set to resemble as closely as possible the Galilean transformation to which they must reduce for sufficiently small values of the velocity of S'.
Consider the motion of the origin of S (x=0) as measured in S'.
From (*) we have that it is described by $\frac{x'}{t'} = \frac{-a}{b}$. But we know that as measured in S' the origin of S is moving with velocity $\frac{x'}{t'} = -v \implies \frac{b\cancel{a}}{a\cancel{b}} = v$

Next consider the description of light ray emitted from the origin at time $t=t'=0$, i.e when the origins of S and S' coincide. Since the speed of light is the same in all inertial frames we have that the light signal is described by

$$x = ct \qquad ; \qquad x' = ct'$$

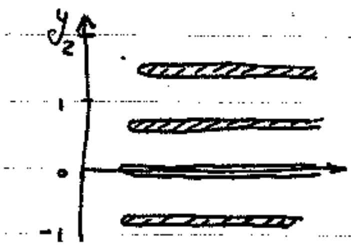
Substituting these expressions in the transformation equations ⊕ we obtain

$$\begin{aligned} ct &= (ac+b)t' \\ ct' &= (ac-b)t \end{aligned} \Rightarrow c^2 = (ac)^2 - b^2 \Rightarrow c^2 = a^2(c^2-v^2)$$

using $b = av$

Therefore $a = \gamma(v) \equiv \dfrac{1}{\sqrt{1-\frac{v^2}{c^2}}}$.

It is also not difficult to conclude that $y' = y$, $z' = z$. If it were not true we could have detect absolute motion. Assume that we place paint brushes each 1 meter along the y and y' axes when S and S' are at rest with ~~each of~~ respect to each other. We then set S' in motion and observe the paint marks the brushes leave on S and S'. If we see a result like



we will have to conclude that not only that the 1 meter intervals in S' have shrunk as measured in S

but also that the 1 meter intervals in S appear larger than 1 meter in S'. But this would mean an asymmetry depending on the direction of motion of the brushes that leave the mark. This would violate our ideas of relativity and homogeneity and isotropy of space.

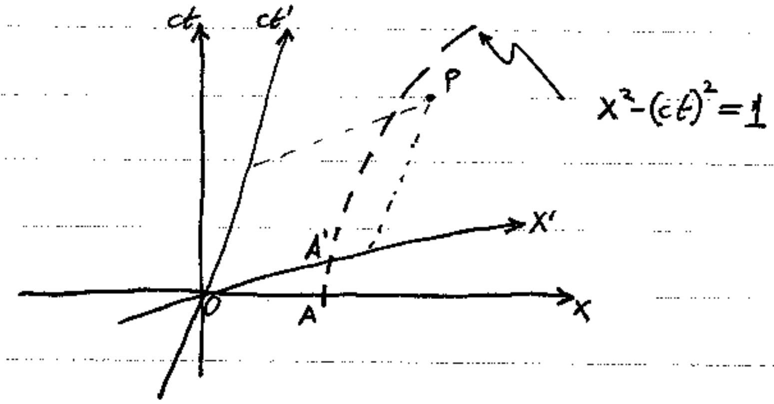
Thus we arrive at the following transformation law:

$$\boxed{\begin{array}{ll} x' = \gamma(x - vt) & x = \gamma(x' + vt') \\ y' = y & y = y' \\ z' = z & z = z' \\ t' = \gamma\left(t - \frac{v}{c^2}x\right) & t = \gamma\left(t' + \frac{v}{c^2}x'\right) \\ \gamma = \frac{1}{\sqrt{1-\left(\frac{v}{c}\right)^2}} & \end{array}}$$

where we obtained the expression for t and t' by solving the two equations $x' = \gamma(x - vt)$, $x = \gamma(x' + vt')$.

These are the Lorentz transformation equations. They were introduced by Lorentz in 1904 as a basis for modifying electromagnetic theory to reconcile the null result of the Michelson Morley experiment with the assumption on the existance of the ether. Einstein discovered these equations a year later following his conviction that Maxwell's equations were correct and that the universe is truely relative.

__Pay attention:__ when considering a Minkowski diagram you can read off the space and time coordinates of a given point event P by drawing lines through it parallel to the space and time axes of the chosen frame and read off the intercepts.



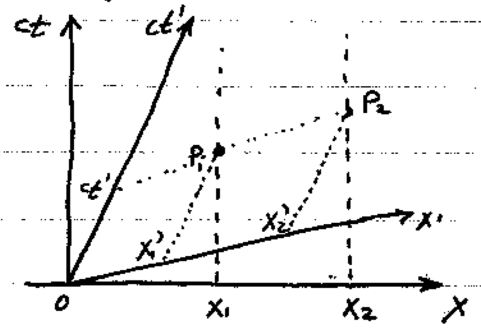
However the __scale__, representing the unit distance, is not the same along x and x'. For example consider the hyperbola $x^2-(ct)^2=1$. Its interception with the x axis gives us the point A corresponding to unit length in frame S. Using the Lorentz transformation it is easy to show that this hyperbola has the same form in terms of the primed coordinates, namely $x^2-(ct)^2 = x'^2-(ct')^2=1$. Thus the intersection of the hyperbola with the x' axis gives us the point A' corresponding to unit length in S'. Note that $OA \neq OA'$
$\Rightarrow$ The x and x' axes are scaled differently.

__Consequences of the Lorentz transformation:__

__Lorentz Contraction:__ What is the __length__ of an object as measured in a frame S? This is the distance between the two ends of the body __measured at the same time in S.__

Consider a body at rest in S. The events corresponding to its ends always have the same space coordinates say x_1 and x_2. Its length is thus

$$l_0 = x_2 - x_1$$



The length of an object in its rest frame is called the Proper Length

In order to determine the length of the object in S' we should measure the positions of its ends at the same time as measured in S', for example the points P_1 and P_2. The length l in S' is then

$$l = x_2' - x_1'$$

Let us transform the coordinates of the events P_1 and P_2 from S' to S: Using the Lorentz transformation we have

$$\begin{aligned} x_1 &= \gamma(x_1' + vt') \\ x_2 &= \gamma(x_2' + vt') \end{aligned} \implies x_2 - x_1 = \gamma(x_2' - x_1')$$

$$\implies l_0 = \gamma l \quad \text{or} \quad l = \frac{l_0}{\gamma} = l_0 \cdot \sqrt{1 - \left(\frac{v}{c}\right)^2}$$

Since γ is always larger than 1 an observer in S will measure a shorter length than the proper length of the body: Lorentz contraction.

Time Dilation:

Consider a clock at rest in point x_0 in frame S. Consider two events corresponding to two different time readings of the clock:

$$\text{event } 1: \ (x_0, t_1) \qquad \qquad \text{event } 2: \ (x_0, t_2)$$

Let us now calculate the coordinates of these events in S'. Using Lorentz transformation we get

$$t_1' = \gamma\left(t_1 - \frac{v}{c^2}x_0\right) \qquad \qquad t_2' = \gamma\left(t_2 - \frac{v}{c^2}x_0\right)$$

$$\Rightarrow \quad t_2' - t_1' = \gamma(t_2 - t_1)$$

If we denote by τ the time difference measured in S' between the events ~~in frame S (τ is called the Proper Time) we get~~
we get

$$\tau = \gamma\,\tau_0 = \frac{1}{\sqrt{1-\left(\frac{v}{c}\right)^2}}\,\tau_0$$

Since γ is always larger than 1 the time difference measured in S' is always <u>larger</u> than the time difference τ_0 measured in the rest frame of the clock: <u>the Proper Time Difference</u>.

<u>Note</u> The time difference as measured in S' is made out of readings of <u>2 different</u> clocks while in S it's the same clock!