# Lecture 2

As we saw the properties of wave propagation are determined by the local oscillators and their interactions. It is thus reasonable to start by studying the ~~prop~~ behavior of a single oscillator.
Before we do that we will spend some time to aquire new mathematical tools that will simplify the description of oscillators and will make the algebra easier.

The new tool is Complex Numbers (F-I Ch. 22)

The Starting Point is the Definition $i = \sqrt{-1}$

Let's find some of the properties of this new number:

Powers:
$$i^2 = (\sqrt{-1})^2 = -1$$
$$i^3 = i^2 \cdot i = -1 \cdot i = -i$$
$$i^4 = i^2 \cdot i^2 = (-1)\cdot(-1) = 1$$
$$i^5 = i^4 \cdot i = i$$
and we start again!

⇒ so with 4 powers the operation ~~repeats~~ repeats itself

Division
$$\frac{1}{i} = \frac{i}{i \cdot i} = \frac{i}{-1} = -i$$
⇒ we can move an $i$ from numerator to denominator with a minus sign.

You all know what real numbers are (something that do not involve an $i$) : $\sqrt{2}$, $-\pi$, $10.23$ ...

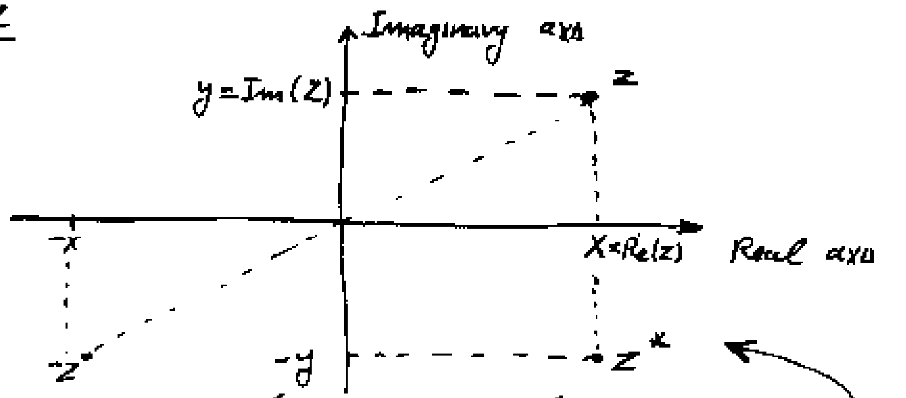
An Imaginary Number is obtained by multiplying a real number by $i$, e.g. $14i$, $-2i$, $\pi i$ ...

A Complex Number is a sum of real and imaginary numbers:

$$z = \underbrace{x}_{\text{real}} + i\underbrace{y}_{\text{real}}$$

We say that $x$ is the Real Part of $z$ and denote it by $\mathrm{Re}\, z = x$ and that $y$ is the Imaginary part of $z$ $\quad \mathrm{Im}\, z = y$

Complex Numbers have Cartesian Representation as points in the Complex Plane




Lets define some more operations on complex numbers

Complex conjugation $\quad$ If $z = x + iy$ then $z^* = x - iy$ is the complex conjugate of $z$

A double conjugation gets you back to the starting point:

$$(z^*)^* = (x-iy)^* = x+iy = z$$

Geometrically conjugation is a reflection across the real axis and double reflection brings you back to the same place.

Don't confuse with: $-z = -x-iy$ which is the image across the origin.

Addition $\quad z_1 = x_1 + iy_1$
$\quad z_2 = x_2 + iy_2 \quad \Rightarrow \quad z_1+z_2 = (x_1+x_2) + i(y_1+y_2)$

(you just add the real and imaginary parts separately).

Multiplication $\quad z_1 \cdot z_2 = (x_1+iy_1)(x_2+iy_2) =$
$= x_1x_2 + ix_1y_2 + iy_1x_2 + (i)^2 y_1y_2$
$= (x_1x_2 - y_1y_2) + (x_1y_2 + y_1x_2)i$

$\Rightarrow \quad \dfrac{z+z^*}{2} = x = \mathrm{Re}\, z$

$\dfrac{z-z^*}{2} = iy = i\,\mathrm{Im}\, z$

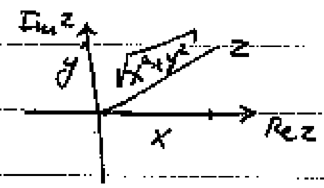
give you a procedure to obtain the real and imaginary parts

$\hookrightarrow \mathrm{Im}\, z = -\dfrac{i}{2}(z-z^*)$

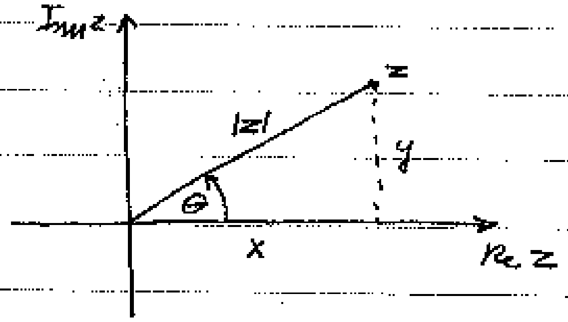
$$z \cdot z^* = [x\cdot x - y(-y)] + [\underbrace{x(-y) + y\cdot x}_{=0}]\,i = x^2+y^2$$

So the distance from the origin - the ~~Amplitude~~ Magnitude of a complex number is

$$|z| = \sqrt{zz^*} = \sqrt{x^2+y^2}$$




Another <u>extremely</u> useful representation of complex numbers is the <u>Polar Representation</u>



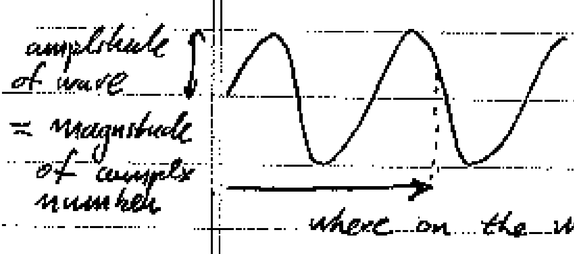

$x = \mathrm{Re}\, z = |z|\cos\theta$

$\Rightarrow y = \mathrm{Im}\, z = |z|\sin\theta$

$\theta$ is measured from the real axis in the "mathematical" sense $= \arctan\left(\frac{y}{x}\right)$

$$\Rightarrow z = |z|\cos\theta + i|z|\sin\theta$$
$$= |z|(\cos\theta + i\sin\theta)$$

So a complex number can be represented as a magnitude and a phase. This is the thing that makes them so convenient in representing oscillatory motion. We will see why exactly next time. Roughly:



A ~~very~~ powerful identity ~~that~~ is

Euler's Identity $\quad e^{i\theta} = \cos\theta + i\sin\theta$

Proof : By using the ~~po~~ Taylor expansions for $e^x$, $\cos x$, $\sin x$:

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$ : Can be taken as the def. of $e^x$

Now let us extend this definition for complex numbers and in particular we find

$$e^{i\theta} = \sum_{n=0}^{\infty} \frac{(i\theta)^n}{n!} = 1 + \frac{i\theta}{1!} + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^3}{3!} + \frac{(i\theta)^4}{4!} + \frac{(i\theta)^5}{5!} + \cdots$$

$$= 1 + i\theta - \frac{\theta^2}{2!} - \frac{i\theta^3}{3!} + \frac{\theta^4}{4!} + \frac{i\theta^5}{5!} + \cdots$$

$$= \left[1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \cdots\right] + i\left[\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \cdots\right]$$

$$= \underbrace{\sum_{n=0}^{\infty} \frac{(-1)^n \theta^{2n}}{(2n)!}}_{\substack{\text{series for} \\ \cos\theta}} + i \underbrace{\sum_{n=0}^{\infty} \frac{(-1)^n \theta^{2n+1}}{(2n+1)!}}_{\substack{\text{series for} \\ \sin\theta}}$$

$$\Rightarrow \quad z = |z|\left(\cos\theta + i\sin\theta\right) = |z|e^{i\theta}$$

The exponential representation makes life easier as we will see

$$z^* = |z|^*(e^{i\theta})^* = |z|(\cos\theta - i\sin\theta) = |z|e^{-i\theta}$$

since $|z|$ is a real number — check.

$$zz^* = |z|e^{i\theta}|z|e^{-i\theta} = |z|^2$$

If $z_1 = |z_1|e^{i\theta_1}$, $z_2 = |z_2|e^{i\theta_2}$ then multiplication is easy

$$z_1z_2 = |z_1|e^{i\theta_1}|z_2|e^{i\theta_2} = |z_1||z_2|e^{i(\theta_1+\theta_2)}$$

multiply the magnitudes — add the phases

hard to ~~do~~ do in cartesian rep. easy here

But $z_1 + z_2 = |z_1|e^{i\theta_1} + |z_2|e^{i\theta_2}$ no easy way to do this one, while easy in Cartesian rep