

## Lecture 18

I ended last lecture by telling you that geometrical optics is the short wavelength limit of wave optics. We saw how it works in one particular example.

Today I would like to derive for you, using wave physics, Snell's law of refraction and show you how to calculate the index of refraction from a simple microscopic model.

When light is passed through material substances its properties can change. In particular one finds that its phase velocity is usually reduced and becomes wave length dependent - light becomes dispersive.

To keep things simple we are going to discuss isotropic materials in which the velocity is independent on the direction of propagation of the wave. We then define the index of refraction as

$$n = \frac{c}{v_p}$$

← phase velocity in vacuum →  
← phase velocity in material →

Note that  $v_p$  and hence  $n$  can be  $\lambda$  dependent.

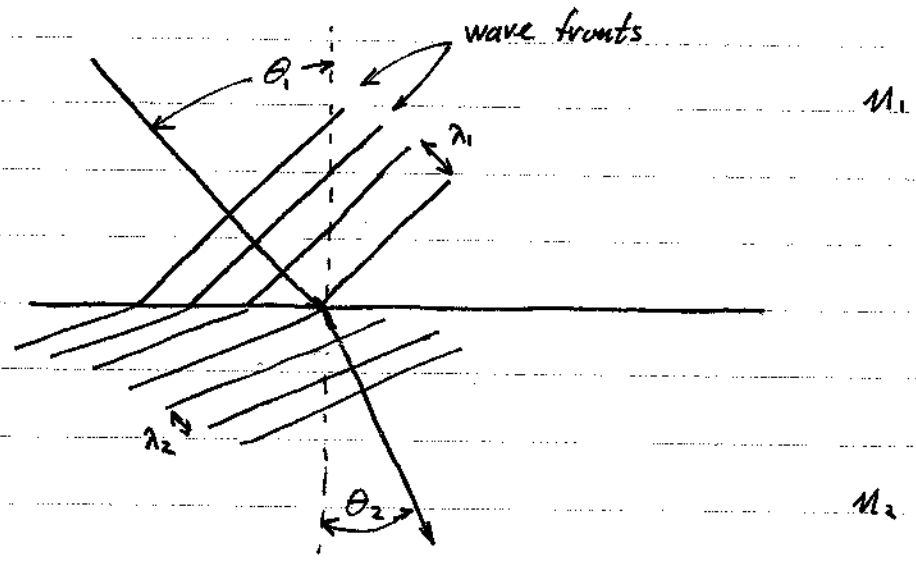
The dispersion relation of the light is now

$$\omega = \frac{c}{n} k \quad \rightarrow \quad \lambda = 2\pi \frac{c}{n \omega}$$

Consider now what happens to a plane wave of wavelength  $\lambda_1$  propagating in a material with index of refraction  $n_1$ , when it encounters a flux interface with material with  $n_2$ .

Let  $\theta_1$  be the angle that the wave vector makes with the normal to the interface

assume  $n_2 > n_1$ :

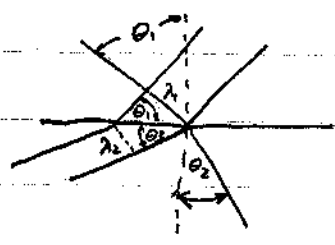


On the interface the fields should be continuous. This should be true for all times and thus it's clear that the waves on both sides of the interface have the same frequency. But if  $\omega_1 = \omega_2 \equiv \omega$

according to our assumption.

$$\frac{\lambda_2}{\lambda_1} = \frac{n_1}{n_2} < 1$$

⇒ The wave fronts in region 2 are separated by a shorter distance. The only way to match the wave fronts on the interface is to tilt the wave vector further towards the normal:



We need:  $\frac{\lambda_1}{\sin \theta_1} = \frac{\lambda_2}{\sin \theta_2}$

Snell's law

$$\Rightarrow \frac{\lambda_1}{\sin \theta_1} = \lambda_1 \frac{n_1}{n_2} \frac{1}{\sin \theta_2} \Rightarrow n_1 \sin \theta_1 = n_2 \sin \theta_2$$

At this point I would like to remind you of the phenomenon of Total Internal Reflection: This happens when  $n_1 > n_2$  for example when light is passed from glass to air. Then

$$\sin \theta_2 = \frac{n_1}{n_2} \sin \theta_1$$

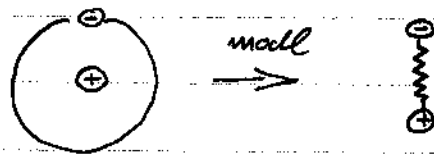
Since  $|\sin \theta| \leq 1$  and  $\frac{n_1}{n_2} > 1$  then for  $\sin \theta_1 > \frac{n_2}{n_1}$  the r.h.s of the above equation is larger than 1 and there is no real solution for  $\theta_2$ . What happens is that for angles above a critical angle  $\theta_c$  for which  $\sin \theta_c = \frac{n_2}{n_1}$  the wave is totally reflected at the interface.

This is the way optical fibers work.



I would like to conclude this part of the course by showing you how to derive an expression for the refraction index.

We'll use a model of an atom or a molecule in which the electrons are bound with force proportional to their displacement.



We will also include the possibility of a damping force in the atomic oscillators. We'll assume that it's proportional to the electron velocity. If we now shine light on these oscillators they will start to oscillate in the electric field of the wave.

The equation of motion is  $\omega_0$  the natural frequency of the oscillator

$$F = eE = m(\ddot{x} + \delta\dot{x} + \omega_0^2 x) \quad \text{with } E = E_0 e^{i\omega t}$$

This is just the equation for a driven oscillator in the presence of damping that we solved at the beginning of the course.

We found

$$x = \frac{e/m}{-\omega^2 + i\delta\omega + \omega_0^2} E_0 e^{i\omega t}$$

For the velocity we have

$$\dot{x} = \frac{i e/m \omega}{-\omega^2 + i\delta\omega + \omega_0^2} E$$

Assuming that the material is isotropic we can generalize this to any direction and get

$$\vec{v} = \frac{i \frac{e}{m} \omega}{-\omega^2 + i\delta\omega + \omega_0^2} \vec{E}$$

If the density of the oscillators = atoms is  $\rho$  the current density associated with the motions of the electrons is  $\vec{J} = e\rho\vec{v}$ .

Further more if  $\omega$  since  $\vec{E} = -\vec{\nabla}\phi - \frac{1}{c} \frac{\partial \vec{A}}{\partial t}$  and if we take  $\phi = 0$  and assume that  $\vec{A} \sim e^{i\omega t}$  we get  $\vec{E} = -\frac{i\omega}{c} \vec{A}$

$$\Rightarrow \vec{J} = \frac{\frac{e^2}{m} \rho \omega^2}{-\omega^2 + i\delta\omega + \omega_0^2} \vec{A}$$

Let's go back now to Maxwell's equations for the potentials.

We now have to solve

$$\begin{aligned} \nabla^2 \vec{A} - \frac{1}{c^2} \frac{\partial^2 \vec{A}}{\partial t^2} &= -\frac{4\pi}{c} \vec{J} \\ &= -\frac{\frac{4\pi}{c} \frac{e^2}{m} \rho \omega^2}{-\omega^2 + i\delta\omega + \omega_0^2} \vec{A} \end{aligned}$$

Looking for a solution in the form  $A e^{i(kx + \omega t)}$  we get the following dispersion relation:

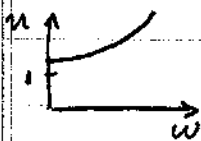
$$\begin{aligned} -k^2 + \frac{\omega^2}{c^2} + \frac{\frac{4\pi}{c} \frac{e^2}{m} \rho \omega^2}{-\omega^2 + i\delta\omega + \omega_0^2} &= 0 \\ \Rightarrow k &= \frac{1}{c} \underbrace{\left[ 1 + \frac{4\pi \frac{e^2}{m} \rho}{-\omega^2 + i\delta\omega + \omega_0^2} \right]}_{n(\omega)} \omega \end{aligned}$$

Note:

1.  $n(\omega)$  is in general a complex function with  $\text{Im}[n(\omega)] \propto \delta$

In that case it means that for real  $k$ ,  $\omega$  has an imaginary part and the wave decays (or is absorbed) as it propagates through the material since  $e^{i\omega t} = e^{-\text{Im}(\omega)t} \cdot e^{i\text{Re}(\omega)t}$

2.  $n$  is a function of the frequency or wavelength



For example for light with  $\lambda_0 = 5890 \text{ \AA}$  in vacuum

$n$	medium
1.33	water
1.36	Alcohol
1.46	Crown glass
2.42	Diamond