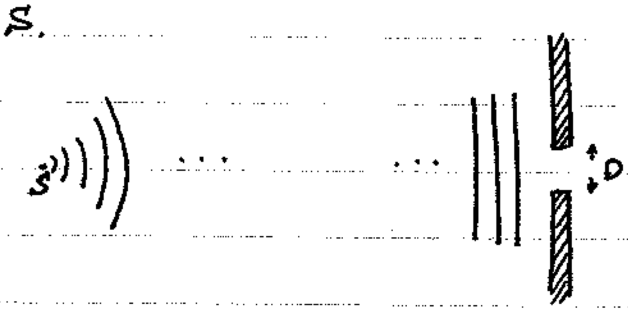
Lecture 17

When the descrete set of sources, that we considered last time, is replaced by a continuous distribution the phenomenon of interference is reffered to as Diffraction.

We would like to calculate the diffraction pattern of a slit of width D in an opaque screen when it is illuminated by a distant point source S.



The EM field of the wave causes the charges on the screen to move. They in turn produce electric and magnetic fields that should be added to the incident wave in order to find the total EM field. This is a difficult task and we're going to make few approximations.

Let us denote by a, b and p the material in the screen above, below and in the slit. Consider now the screen before p was removed to form the slit. The total field behind the screen, which is zero, is a superposition of the field emitted by S and by a, b and p. Thus <u>before</u> removing p the total field behind the screen is

$$E_S + E_a + E_b + E_p = 0$$

Now remove p and <u>assume</u> that the motion of the charges in a and b is not affected by it. This is an <u>approximation</u> since these charges are driven by the total fields at their sites, and that includes the fields radiated by the electrons that were in p.

Under this approximation the fields E_a', E_b' emitted by a and b after the removal is $E_a' \approx E_a$, $E_b' \approx E_b$ and we have for the field behind the screen

$$E = E_a' + E_b' + E_s \approx E_a + E_b + E_s = (E_a + E_b + E_p + E_s) - E_p = -E_p$$

$\Rightarrow$ The remaining field is the same (up to a sign) as that which was being emitted by p when it was in place.

In the case where the incident plane wave is propagating perpendicular to the screen all the electrons in p oscillate in phase and with the same amplitude A.

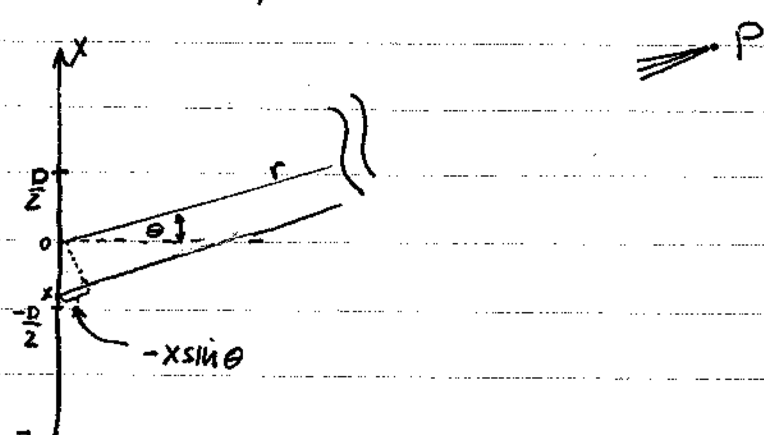
We then have for the field at a distant point P behind the screen

$$E = \sum_{i \,\leftarrow\, \text{electrons in } p} E_i$$

$$= \sum \Delta x \cdot \rho \cdot E$$

# of electrons per unit length in the x direction

field due to electrons around x.



$$= \frac{A\rho}{r} \int_{-\frac{D}{2}}^{\frac{D}{2}} dx\, e^{i\left[k(r - x\sin\theta) - \omega t\right]}$$

$$= AP\frac{1}{r}e^{i(kr-\omega t)}\boxed{\int_{-D/2}^{D/2} e^{-ikx\sin\theta}\,dx}$$

← The Fourier transform of the slit transmission function

$$f_s = \begin{cases} 1 & |x| < D/2 \\ 0 & |x| > D/2 \end{cases}$$

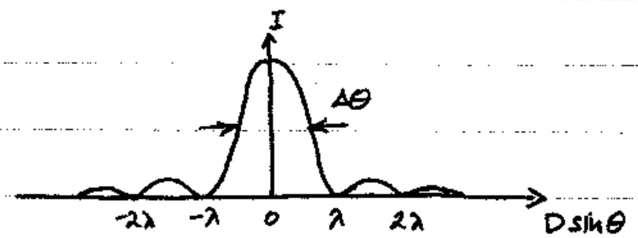
with respect to $-k\sin\theta$

$$= \frac{AP}{r}e^{i(kr-\omega t)}\left.\frac{1}{-ik\sin\theta}e^{-ikx\sin\theta}\right|_{-D/2}^{D/2}$$

$$= \frac{2AP}{r}e^{i(kr-\omega t)}\frac{\sin\left(k\frac{D}{2}\sin\theta\right)}{k\sin\theta}$$

$$= \frac{APD}{r}e^{i(kr-\omega t)}\frac{\sin\left(\pi \frac{D}{\lambda}\sin\theta\right)}{\pi\frac{D}{\lambda}\sin\theta}$$

$$\Rightarrow I \propto E^2 \sim \frac{\sin^2\left(\pi\frac{D}{\lambda}\sin\theta\right)}{\left(\pi\frac{D}{\lambda}\sin\theta\right)^2}:$$



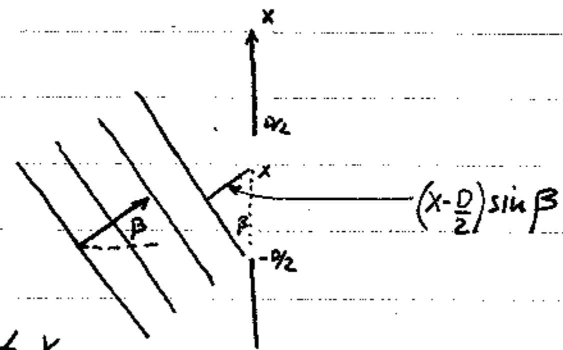
There are 2 general lessons to be learned from this result:

1. The diffraction field is proportional to the Fourier transform of the transmission function (the intensity to its square).
The transmission function is in general a complex function with an amplitude which describes absorption and a phase that corresponds to the phase shift induced by the system.

2. The angular width $\Delta\theta$ of the diffraction pattern = the angular width of the central peak $\sim \frac{\lambda}{D}$ (we are dealing with small angles so $\sin\theta \sim \theta$)

To further illustrate these points consider the same slit but now with a plane wave incident at an angle β to the normal

Now the charges in ρ oscillate with different phases.
If the phase is 0 at $x = -\frac{D}{2}$ then it is $k\left(x - \frac{D}{2}\right)\sin\beta$ at point x.

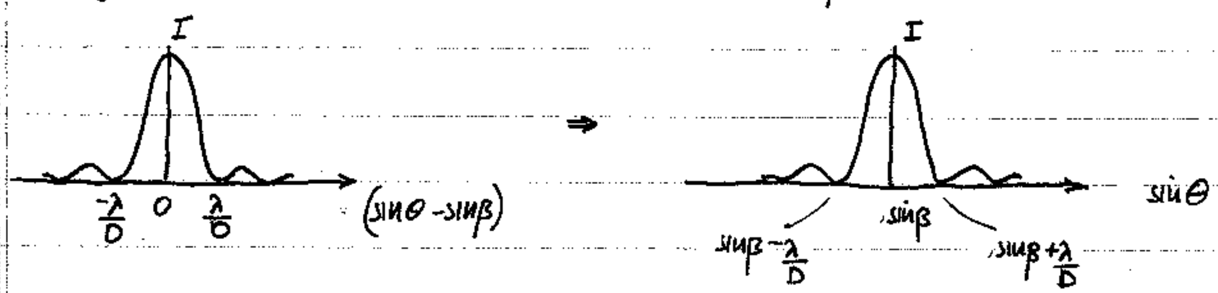


These are just the phases of the wave when it hits the point $-\frac{D}{2}$ and x.
We then have

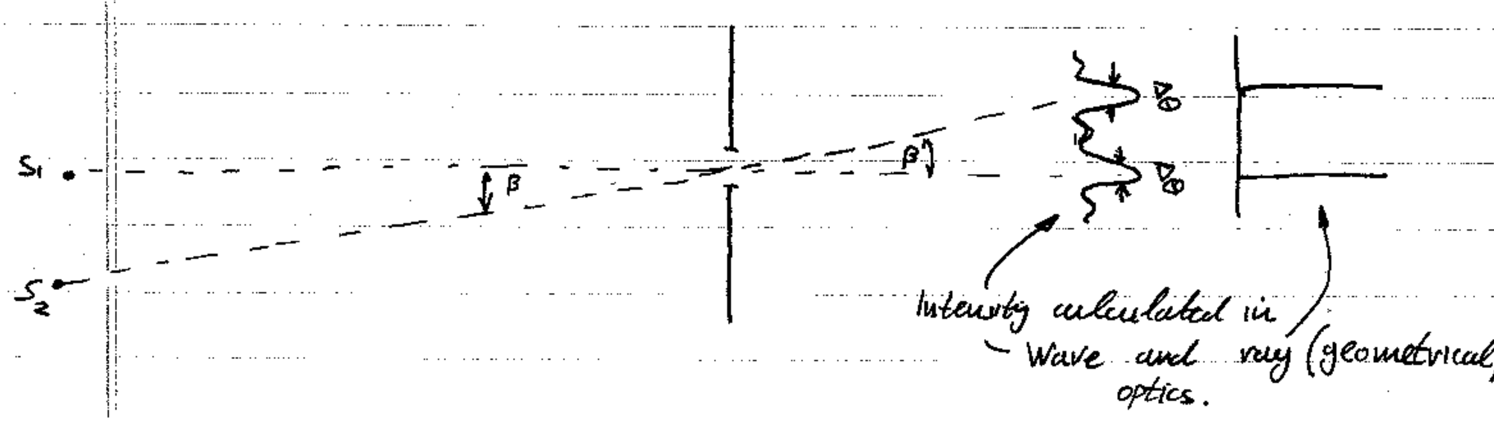
$$E = \frac{A\rho}{r} e^{i\left[k\left(r - \frac{D}{2}\sin\beta\right) - \omega t\right]} \int_{-D/2}^{D/2} e^{-ikx(\sin\theta - \sin\beta)}\, dx$$

$\Rightarrow$ We get the same result but with the replacement $\sin\theta \rightarrow \sin\theta - \sin\beta$

$\Rightarrow$



We are dealing with small angles $\Rightarrow$ $\sin\theta \sim \theta$ $\sin\beta \sim \beta$ and the image is shifted by an angle β.
If we neglect interference between the patterns of the 2 sources we get:

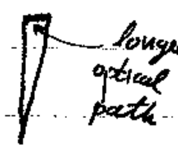


The <u>Rayleigh Criterion</u> states that the two sources can be just resolved when the maximum of the intensity from S_1 fall on the first minimum of the intensity pattern of the other point source.
That is when

$$\sin\beta \simeq \beta = \frac{\lambda}{D}$$

$\Rightarrow$ The larger D is and the smaller λ is the greater is our ability to resolve the sources.

One can achieve a similar shift in the image position by placing a wedge of clear glass over the slit and use S_1. Since the velocity of light in glass is smaller than the one in vacuum the wave at the top of the slit acquires a phase shift due to the longer optical path. By choosing an appropriate shape for the wedge one can induce a phase shift of $e^{ikx\sin\beta}$.



The transmission function of the slit + wedge is then

$$f = \begin{cases} e^{ikx\sin\beta} & |x| < \frac{D}{2} \\ 0 & |x| > \frac{D}{2} \end{cases}$$

and the image behind the screen due to the light from S_1 is the F.T. of that.

Finally, geometrical (ray) optics predicts that S_1 and S_2 images behind the screen are sharp points at $\theta = 0$ and $\theta = \beta$. This is the limit $\frac{\lambda}{D} \to 0$ limit of the result we calculated with wave optics.
This is a <u>general statement</u>:
Wave optics reduces to geometrical optics when the typical dimensions of the optical system are much larger than the wavelength of light.