Lecture 16

We will now shift our attention to cases in which waves from several sources add up to give a new "superwave" - we already encountered this phenomenon and named it Interference.
In this context we will often use the concept of a Spherical Wave which is of importance in its own right.

Let us look at the wave equation $\nabla^2\psi - \frac{1}{c^2}\frac{\partial^2\psi}{\partial t^2} = 0$
and look for a solution that depends only on the distance $r = \sqrt{x^2+y^2+z^2}$ from the origin. For such a solution one has

$$\frac{\partial\psi(r)}{\partial x} = \frac{\partial\psi}{\partial r}\frac{\partial r}{\partial x}$$

(in principal there are additional terms here involving $\frac{\partial\psi}{\partial\theta}$ and $\frac{\partial\psi}{\partial\varphi}$ where θ and φ are the polar and azimuthal angles, but these terms vanish under our assumption)

$$= \frac{x}{r}\frac{\partial\psi}{\partial r}$$

$$\Rightarrow \frac{\partial^2\psi}{\partial x^2} = \frac{1}{r}\frac{\partial\psi}{\partial r} + x\frac{x}{r}\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\psi}{\partial r}\right) = \frac{x^2}{r^2}\frac{\partial^2\psi}{\partial r^2} + \frac{1}{r}\left(1-\frac{x^2}{r^2}\right)\frac{\partial\psi}{\partial r}$$

We obtain similar results for $\frac{\partial^2\psi}{\partial y^2}, \frac{\partial^2\psi}{\partial z^2}$ as the above just replace x by y and z.

$$\Rightarrow \nabla^2\psi = \frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} + \frac{\partial^2\psi}{\partial z^2} = \frac{x^2+y^2+z^2}{r^2}\cdot\frac{\partial^2\psi}{\partial r^2} + \frac{1}{r}\left(3 - \frac{x^2+y^2+z^2}{r^2}\right)\frac{\partial\psi}{\partial r}$$

$$= \frac{\partial^2\psi}{\partial r^2} + \frac{2}{r}\frac{\partial\psi}{\partial r} = \frac{1}{r}\frac{d^2}{dr^2}(r\psi)$$

The wave equation then becomes (after multiplying by r)

$$\frac{\partial^2(r\psi)}{\partial r^2} - \frac{1}{c^2}\frac{\partial^2(r\psi)}{\partial t^2} = 0$$

We know the general solution to this equation, It is:

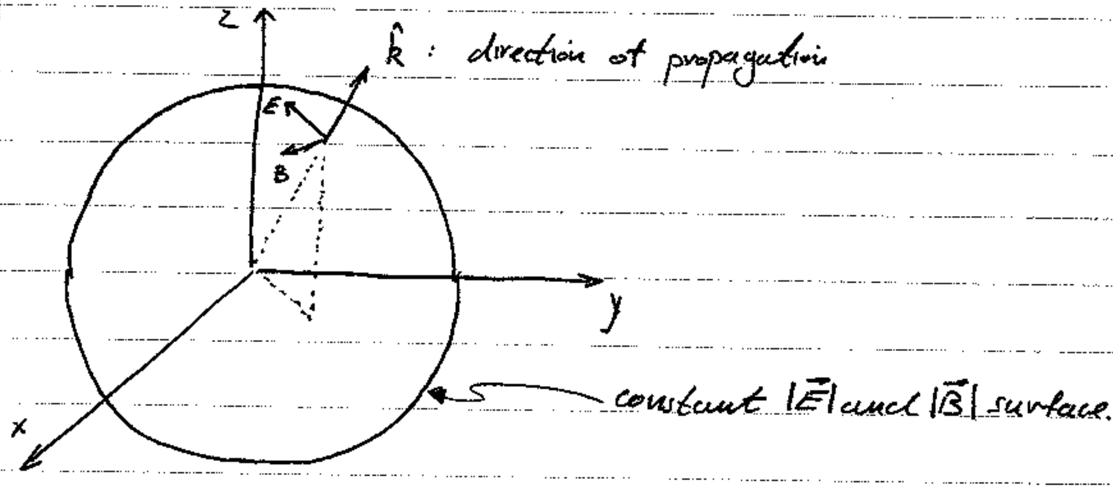
$$r\psi = f(r \pm ct) \quad \Rightarrow \quad \psi = \frac{f(r \pm ct)}{r}$$

The function with the + sign represents a general spherical wave travelling outward from the origin with velocity c. The function with the - sign describes a sperical wave traveling inward from large r toward the origin. We'll concentrate on the former.
The factor $\frac{1}{r}$ says that the amplitude of the wave deminishes in proportion to $\frac{1}{r}$ as the wave propogates out of the origin. This is nothing but energy conservation. We know that the energy current density in a wave depends on the square of its amplitude. As the wave spreads the area through which it passes grows as r^2 in order for the energy flux to remain constant the energy current density must decrease as $\frac{1}{r^2}$ - as indeed is the case.

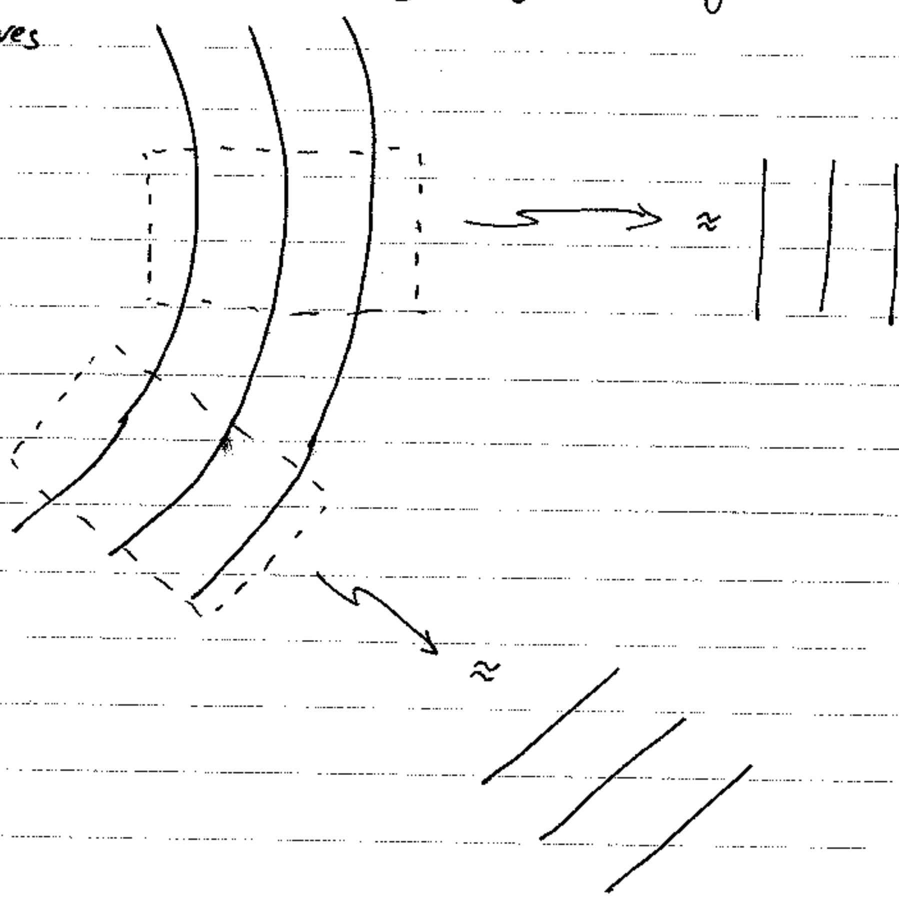
A final point: There is something strange with the solution we found: It goes to infinity at the origin. What really happened is that we have not solved the free wave equation everywhere. We've solved it with zero on the r.h.s except at the origin. Some of the steps in our derivation are not "legal" at $r=0$. The "thing" that lives at the origin and appear on the r.h.s there is the charge that produces the wave.

(can find this charge by considering Gauss law for a sphere around the origin)

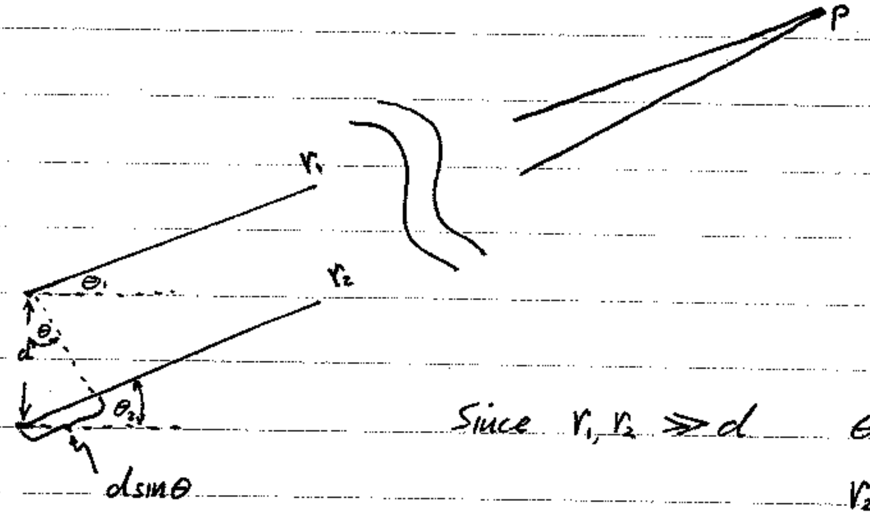
So how does a Spherical, Linearly Polarized EM wave looks like?



If we look at the wave ~~sur~~ fronts far from the origin and restrict our attention to not too large regions, they look locally like plane waves



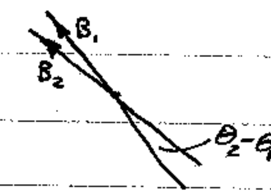
Let us study the interference between spherical waves emitted by 2 point sources separated by distance d.
We will restrict our treatment to the case where the detector is located at a distant point P in the plane containing the two sources.



Since $r_1, r_2 \gg d$ $\quad \theta_1 \simeq \theta_2 \equiv \theta$

$r_2 \simeq r_1 + d\sin\theta$

The spherical waves emitted by S_1 and S_2 can be approximated at the point P by plane waves. According to our approximation we'll neglect the small difference in the direction of the two wave fronts. This means that the direction of the electric and magnetic fields on both fronts is the same and we can add the fields from the two waves like scalars.



Assuming that the amplitudes (A) and frequencies of the waves emitted by S_1 and S_2 are the same (but not necessarely their phases) we have for the field at P:

$$E(r,\theta,t) = \underbrace{\frac{A}{r_1}\cos(kr_1 - \omega t + \varphi_1) + \frac{A}{r_2}\cos(kr_2 - \omega t + \varphi_2)}_{E_1(r,\theta,t)+E_2(r,\theta,t)=}$$

$$\simeq \frac{2A}{r}\cos(kr - \omega t + \varphi_{av})\cos\left(k\frac{r_1 - r_2}{2} + \frac{\varphi_1 - \varphi_2}{2}\right)$$

$r \equiv \frac{r_1 + r_2}{2}$

$\varphi_{av} = \frac{\varphi_1 + \varphi_2}{2}$

$$\simeq \frac{2A}{r} \cos\left(kr - \omega t + \varphi_{av}\right) \cos\left(\frac{\pi d \sin\theta}{\lambda} + \frac{\varphi_1 - \varphi_2}{2}\right)$$

The detector is measuring the intensity, i.e. the time averaged energy flux of the wave. As we saw the intensity is proportional to the square of the field amplitude. We thus have

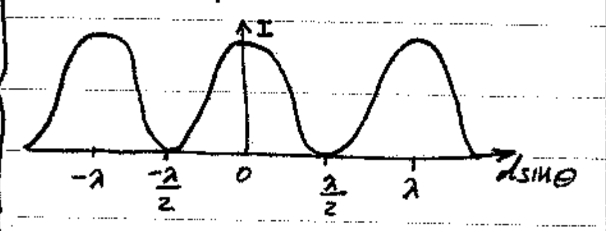
$$I(\theta) = \underbrace{\langle |\vec{S}| \rangle}_{\text{time average}} \propto \langle E^2 \rangle = \frac{4A^2}{r^2} \cos^2\left(\frac{\pi d \sin\theta}{\lambda} + \frac{\varphi_1 - \varphi_2}{2}\right) \underbrace{\langle \cos^2(kr - \omega t + \varphi_{av}) \rangle}_{= 1/2}$$

$$\langle E^2 \rangle = \underbrace{\langle E_1^2 \rangle + \langle E_2^2 \rangle}_{\text{intensity of independent sources}} + \underbrace{2\langle E_1 E_2 \rangle}_{\text{Interference term}}$$

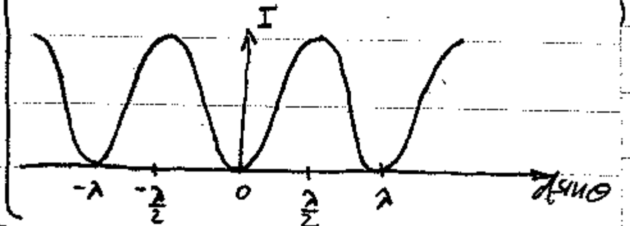
(as we showed last time)

$$\Rightarrow \quad I(\theta) = I_{max} \cos^2\left(\frac{\pi d}{\lambda} \sin\theta + \frac{\varphi_1 - \varphi_2}{2}\right) =$$

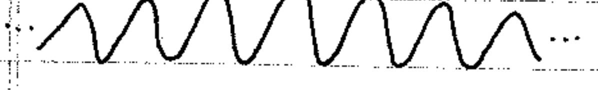
2 sources in phase : $\varphi_1 = \varphi_2$



2 sources out of phase $\varphi_1 = \varphi_2 + \pi$



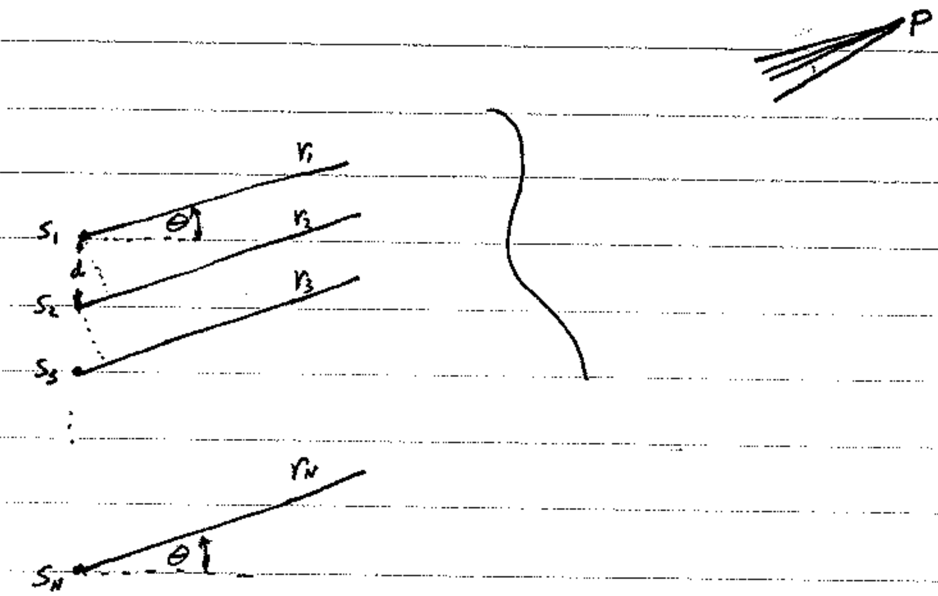
[Note that r is also a function of θ if the detector is place on a flat screen. In that case r increases with θ and so the maximal intensity will also decrease: but very slowly if the screen is far away as we assumed



The positions of the peaks depend on the wave length λ.

$\longrightarrow$ If we change the frequencies of S_1, S_2 the peaks will shift.

Consider now the interference pattern from N point sources.



Again we consider only the far field pattern and accordingly take $\theta_1 = \theta_2 = \cdots = \theta_N \equiv \theta$ and add the fields as scalars. We now have $r_2 = r_1 + d\sin\theta$, $r_3 = r_2 + d\sin\theta = r_1 + 2d\sin\theta$, $\cdots$ $r_N = (N-1)d\sin\theta + r_1$
Assuming the sources are in phase (and taking $\varphi = 0$) and working in the complex notation we have that the field at P is

$$E = \sum_{i=1}^{N} E_i = \frac{A}{r}\sum_{n=1}^{N} e^{i(kr_n - \omega t)} = \frac{A}{r} e^{i(kr-\omega t)} \sum_{n=0}^{N-1} e^{inkd\sin\theta}$$

Denoting $a = e^{ikd\sin\theta}$ we have

$$S \equiv \sum_{n=0}^{N-1} e^{ind\sin\theta} = \sum_{n=0}^{N-1} a^n = 1 + a + a^2 + \cdots + a^{N-1}$$
$$= 1 + a + a^2 + \cdots + a^{N-1} + a^N - a^N$$
$$= 1 - a^N + a\left(1 + a + \cdots + a^{N-1}\right) = 1 - a^N + aS$$

$$\Rightarrow S = \frac{1-a^N}{1-a}$$

$$\Rightarrow \quad S = \frac{1 - e^{iNkd\sin\theta}}{1 - e^{ikd\sin\theta}} = \frac{e^{i\frac{N}{2}kd\sin\theta}}{e^{i\frac{d}{2}k\sin\theta}} \left[\frac{e^{-i\frac{N}{2}kd\sin\theta} - e^{i\frac{N}{2}kd\sin\theta}}{e^{-i\frac{kd}{2}\sin\theta} - e^{i\frac{kd}{2}\sin\theta}} \right]$$

$$= e^{i\frac{N-1}{2}kd\sin\theta} \cdot \frac{\sin \frac{N}{2}kd\sin\theta}{\sin \frac{1}{2}kd\sin\theta}$$

The physical E is then:

$$E = Re\left[\frac{A}{r} \frac{\sin \frac{N}{2}kd\sin\theta}{\sin \frac{1}{2}kd\sin\theta} e^{i\left(kr + k\frac{N-1}{2}d\sin\theta - \omega t\right)} \right]$$

$$= \frac{A}{r} \frac{\sin \frac{N}{2}kd\sin\theta}{\sin \frac{1}{2}kd\sin\theta} \cos\left[k\left(r + \frac{N-1}{2}d\sin\theta\right) - \omega t \right]$$

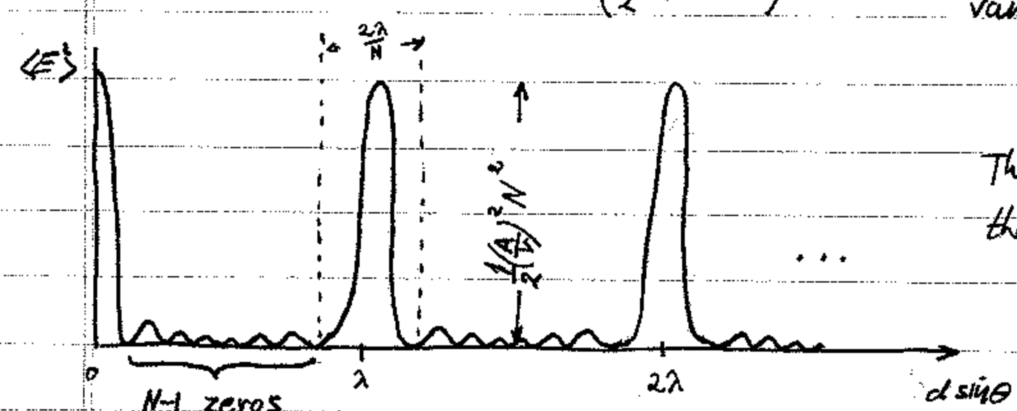
Averaging the intensity over time we find the interference intensity pattern: Using $\langle \cos^2(kx - \omega t)\rangle = \frac{1}{2}$

$$\langle I \rangle \propto \langle E^2 \rangle = \frac{1}{2}\left(\frac{A}{r}\right)^2 \frac{\sin^2\left(\frac{N}{2}kd\sin\theta\right)}{\sin^2\left(\frac{1}{2}kd\sin\theta\right)}$$

large maxima: when denominator vanishes: $\frac{1}{2}kd\sin\theta = n\pi$

$\Rightarrow \; d\sin\theta = n\lambda$

The numerator vanishes there but the ratio $\to N^2$



The peaks get sharper and higher as we increse N.

The angles for which the high maxima occur are those for which the path length increment between consequtive sources: $d\sin\theta$ is $0, \pm\lambda, \pm 2\lambda$ etc, corresponding to complete Constructive Interference between all sources.
These are called the Principal Maxima.
The maximum at $\theta = 0$ is called the Central or Zeroth Order Maximum. Those with $n = \pm 1$ are called first order etc.
The central maximum is special since its position is independent of the wave length. Thus for a white light the central maximum is white while the other maxima occur at different angles for different wave length and appear colorful and blurry.