Lecture 15   Reading Assignment: Interference and Diffraction: F-I Ch 29,30
W- 9.2, 9.3, 9.6

Energy Conservation and Electromagnetism

When we discussed the continuity equation I stressed the fact that one can write a similar equation for any conserved quantity. If we then denote by u the Energy Density of the EM field (the analog of the charge density in the case we discussed before) and by $\vec{S}$ the Energy Current Density or the Energy Flux we have, Provided the EM energy is conserved, that

$$\frac{\partial u}{\partial t} = -\vec{\nabla}\cdot\vec{S}$$

But is the EM energy always conserved? The answer is of course No. It's the total energy which is conserved. The EM field can push charges around thus transforming EM energy into kinetic and potential energy of the charges.
Thus if we consider the EM energy in a volume V it decreases

$$-\frac{\partial}{\partial t}\int_V u\, dv = \int_A \vec{S}\cdot d\vec{a} + \text{Work done on matter inside } V \text{ per unit time.}$$

(either because field energy is flowing out of the volume   or because the field loses energy to matter)

You know that if a force $\vec{F}$ acts on a particle moving with velocity $\vec{v}$ the energy of the particle changes by $\vec{F}\cdot\vec{v}$ per unit time. This is the Power = Work done per unit time.

The rate at which the EM field does work on a particle with charge q is then

$$q\left(\vec{E} + \frac{\vec{v}\times\vec{B}}{c}\right)\cdot\vec{v} = q\vec{v}\cdot\vec{E}$$

The magnetic field <u>does not</u> do work on the charge since ~~its direction~~ the direction of the force that it exerts is always perpendicular to the velocity of the charge.

Therefore the work per unit time that the EM field do on the charges inside V is $\int_V \underbrace{(\rho\vec{v})}_{\text{charge density}}\cdot\vec{E}\, d^3v = \int_V \vec{E}\cdot\underbrace{\vec{J}}_{\text{current density}}\, d^3v$

Using Gauss theorem we then have

$$-\frac{\partial}{\partial t}\int_V u\, d^3v = \int_V \vec{\nabla}\cdot\vec{S}\, d^3v + \int_V \vec{E}\cdot\vec{J}\, d^3v \Rightarrow -\frac{\partial u}{\partial t} = \vec{\nabla}\cdot\vec{S} + \vec{E}\cdot\vec{J}$$

We would like to find expressions for u and $\vec{S}$ we will do it by trying to write $\vec{E}\cdot\vec{J}$ as the sum of a time derivative of some scalar field and ~~a~~ divergence of a vector field. We'll then identify the former with u and the latter with $\vec{S}$.

Using Maxwell's fourth equation we got

$$\vec{E}\cdot\vec{J} = \vec{E}\cdot\left[\frac{c}{4\pi}\vec{\nabla}\times\vec{B} - \frac{1}{4\pi}\frac{\partial\vec{E}}{\partial t}\right]$$

We now use the following equalities:

1. $\frac{\partial E^2}{\partial t} = \frac{\partial}{\partial t}(\vec{E}\cdot\vec{E}) = 2\vec{E}\cdot\frac{\partial \vec{E}}{\partial t} \quad \Rightarrow \quad \vec{E}\cdot\frac{\partial \vec{E}}{\partial t} = \frac{1}{2}\frac{\partial E^2}{\partial t^2}$

2. $\vec{\nabla}\cdot(\vec{E}\times\vec{B}) = (\vec{\nabla}\times\vec{E})\cdot\vec{B} - \vec{E}\cdot(\vec{\nabla}\times\vec{B})$: see F II 27.4 for details

$$= -\frac{1}{c}\frac{\partial \vec{B}}{\partial t}\cdot\vec{B} - \vec{E}\cdot(\vec{\nabla}\times\vec{B})$$ using Faraday's law

$$= -\frac{1}{2c}\frac{\partial B^2}{\partial t} - \vec{E}\cdot(\vec{\nabla}\times\vec{B})$$: same way as 1

$$\Rightarrow \vec{E}\cdot(\vec{\nabla}\times\vec{B}) = -\frac{1}{2c}\frac{\partial B^2}{\partial t} - \vec{\nabla}\cdot(\vec{E}\times\vec{B})$$

We then find

$$\vec{E}\cdot\vec{J} = -\frac{c}{4\pi}\vec{\nabla}(\vec{E}\times\vec{B}) - \frac{1}{8\pi}\frac{\partial}{\partial t}(E^2+B^2)$$

Comparing this to what we want $\vec{E}\cdot\vec{J} = -\frac{\partial u}{\partial t} - \vec{\nabla}\cdot\vec{S}$ we identify

$$\boxed{u = \frac{1}{8\pi}(E^2+B^2) \qquad \vec{S} = \frac{c}{4\pi}(\vec{E}\times\vec{B})}$$

units in CGS $\left[\frac{\text{erg}}{\text{cm}^3}\right]$

This vector is called the <u>Poynting Vector</u>

units in CGS $\left[\frac{\text{erg}}{\text{cm}^2\,\text{sec}}\right]$

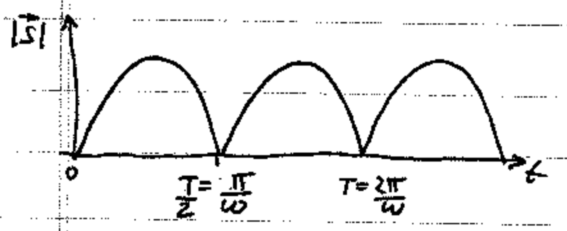
If we thing about the EM energy as a fluid with density u it gives us the current density of that fluid. Its flux through a surface A is the change of the EM energy inside A <u>provided</u> there are no free charges there. Its direction is the direction of the flow of EM energy.

Let's calculate the Poynity vector for the EM plane wave:

$$\vec{S} = \frac{c}{4\pi}\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & E_0 \sin(kx-\omega t) & 0 \\ 0 & 0 & E_0 \sin(kx-\omega t) \end{vmatrix} = \frac{c}{4\pi} E_0^2 \sin^2(kx-\omega t)\, \hat{i}$$

Things to notice:
1. The energy flux is proportional to the $(\text{field})^2$
2. $\vec{S}$ is directed along the direction of propagation of the wave (not always the case, for example waves in dielectrics).

If we look at $|\vec{S}|$ at a point as a function of time we find



averge $\vec{S}$ over time $\Longrightarrow$

$$\langle \vec{S} \rangle = \frac{1}{T}\int_0^T |S|\, dt =$$

$$= \frac{1}{T}\frac{c}{4\pi}\int_0^T E_0^2 \left[\frac{1-\cos[2(kx-\omega t)]}{2}\right] =$$

$$= \frac{1}{T}\frac{c}{4\pi}\left[\frac{E_0^2}{2} t + \underbrace{\frac{\sin 2(kx-\omega t)}{4\omega}}_{\sin 2(kx-2\pi) - \sin 2kx = 0}\right]_0^T = \frac{cE_0^2}{8\pi}$$

$\langle \vec{S} \rangle \sim E^2$ is called the <u>Intensity</u> of the wave

What is the energy density? $u = \frac{1}{8\pi}(E^2+B^2) = \frac{1}{4\pi} E_0^2 \sin^2(kx-\omega t)$

Again averaging over time we find

$$\langle u \rangle = \frac{E_0^2}{8\pi}$$

Remembering the water analogy where $J = v\rho$ we find that we velocity at which the energy is transffered by the wave is $\frac{\langle \vec{S} \rangle}{\langle u \rangle} = c$.

This is the phase and group velocity of the wave. In cases where $v_\varphi \neq v_g$ the energy propagates with v_g as you'll show in your HW.