Lecture 14

Last time we found Maxwell's equations for the scalar potential ϕ and the vector potential $\vec{A}$. Today we will solve them in the vacuum i.e where no charges or currents exist. In this case the equations are simple wave equations:

$$\text{I.} \quad \nabla^2\phi - \frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2} = 0$$

$$\text{II.} \quad \nabla^2\vec{A} - \frac{1}{c^2}\frac{\partial^2\vec{A}}{\partial t^2} = 0$$

and we have to supplement them with the Lorentz condition

$$\vec{\nabla}\cdot\vec{A} = -\frac{1}{c}\frac{\partial\phi}{\partial t}$$

We are going to look for a solution with $\phi = 0$ and assume that it depends only on one coordinate in space, say x. Equation I is then trivially satisfied and the Lorentz condition reads

$$\frac{\partial A_x}{\partial x} = 0 \qquad \left(\text{the } \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} \text{ terms vanish due to our assumption}\right)$$

$\Rightarrow$ A_x can't depend on x it may be a function of t only. Examining the x component of Eq. II we find

$$\underset{\overset{\shortparallel}{0}}{\frac{\partial^2 A_x}{\partial x^2}} - \frac{1}{c^2}\frac{\partial^2 A_x}{\partial t^2} = 0 \quad \Rightarrow \quad A_x = a + bt$$

the general solution to this eq.

Remembering that $\vec{E} = -\vec{\nabla}\phi - \frac{1}{c}\frac{\partial \vec{A}}{\partial t}$ we have $E_x = -\frac{1}{c}\frac{\partial A_x}{\partial t} = -\frac{b}{c}$

This is just a constant time independent electric field in the x direction. We are looking for EM waves so we don't care about this simple solution and will correspondingly set $A_x = 0 \Rightarrow E_x = 0$. (Since A_x is space independent it's clear that it doesn't contribute to $\vec{B} = \vec{\nabla} \times \vec{A}$).

We are then left with the y and z components of II. They are just the familiar wave equations, now for A_y and A_z:

$$\frac{\partial^2 A_y}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2 A_y}{\partial t^2} = 0 \quad , \quad \frac{\partial^2 A_z}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2 A_z}{\partial t^2} = 0$$

By now we know very well what are their solutions:

$$A_y(x,t) = A_y\, e^{i(kx \pm \omega t)} \quad ; \quad A_z(x,t) = A_z\, e^{i(kx \pm \omega t)}$$

Here A_y and A_z are complex numbers. Remember, the physical A_y and A_z are the real parts of the above complex quantities. From the wave equations we find that the dispersion law for these waves is $\boxed{\omega = ck}$: they propagate with phase (and group) velocities equal to the speed of light. Not surprising (to us but not to Maxwell) since we know that light is nothing else than EM waves with particular wavelength.

What are the electric and magnetic fields associated with the waves?

$\phi = 0$

$$E_y = -\frac{1}{c}\frac{\partial A_y}{\partial t} = \mp \frac{i\omega}{c} A_y e^{i(kx\pm\omega t)}$$ and similarly for E_z

$$\vec{B} = \vec{\nabla}\times\vec{A} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & A_y & A_z \end{vmatrix} = \underbrace{\left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right)}_{=0}\hat{i} - \frac{\partial A_z}{\partial x}\hat{j} + \frac{\partial A_y}{\partial x}\hat{k}$$

we took $A_x = 0$

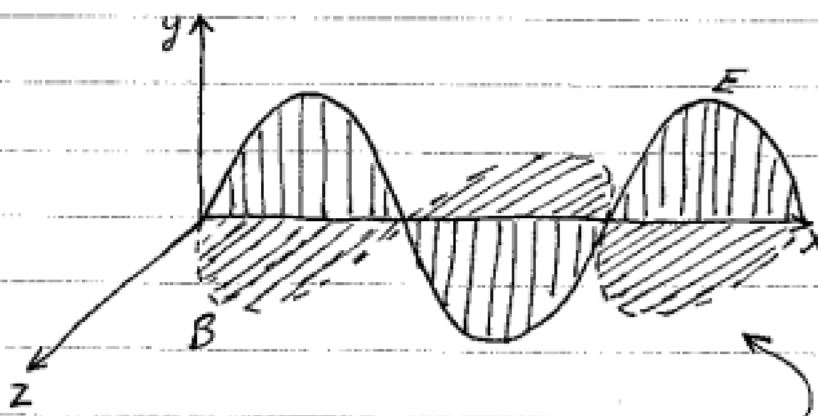
since A_y, A_z independent of y, z.

$$= -ikA_z e^{i(kx\pm\omega t)}\hat{j} + ikA_y e^{i(kx\pm\omega t)}\hat{k}$$

Again the physical $\vec{E}$ and $\vec{B}$ are the real parts of the above. Let's take first $A_z = 0$ and $A_y = -\frac{c}{\omega}E_0$ we then find, for the right propagating $(kx-\omega t)$ wave;

$$E_y = E_0 \sin(kx-\omega t) \quad ; \quad E_x = E_z = 0$$

$$B_z = \frac{ck}{\omega}E_0 \sin(kx-\omega t) = E_0 \sin(kx-\omega t) \; ; \; B_x = B_y = 0$$



$\vec{E}$ field is perp. to $\vec{B}$ field and they both are perp. to the direction of propagation of the wave.

EM waves are Transverse waves

The solution we found is called a <u>Plane Wave</u> since the surfaces of constant $\vec{E}$ and $\vec{B}$ are planes parallel to the y-z plane.

Some typical frequencies and wave lengths of EM waves:

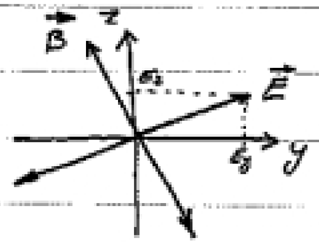
$$\underline{\underline{\lambda = \frac{2\pi}{k}}} \qquad \underline{\underline{f = \frac{\omega}{2\pi} = \frac{ck}{2\pi} = \frac{c}{\lambda}}}$$

FM radio station	~ 10 m	$\sim 10^7$ Hz ← $\text{Hz} = \frac{1}{\text{sec}}$
Microwave oven	~ 10 cm	$\sim 10^9$ Hz
Visible light	3500 – 7500 Å (blue) (red) ← Å $= 10^{-8}$ cm	$5\text{-}10 \cdot 10^{14}$ Hz
X-rays	~ 1 Å	$\sim 10^{18}$ Hz
γ-rays	$\sim 10^{-3}$ Å	$\sim 10^{21}$ Hz

Our solution describes a <u>Linearly Polarized Wave</u> that is the electric field is oscillating in one direction (the y direction for our solution) and the magnetic field oscillates in the perpendicular direction.

We can build a linearly polarized wave where the $\vec{E}$ field oscillate in an arbitrary direction perpendicular to the direction of propagation of the wave:

$$\vec{E} = E_y \sin(kx - \omega t)\,\hat{y} + E_z \sin(kx - \omega t)\,\hat{z} = (E_y \hat{y} + E_z \hat{z}) \sin(kx - \omega t)$$
$$\vec{B} = E_y \sin(kx - \omega t)\,\hat{z} - E_z \sin(kx - \omega t)\,\hat{y} = (-E_z \hat{y} + E_y \hat{z}) \sin(kx - \omega t)$$



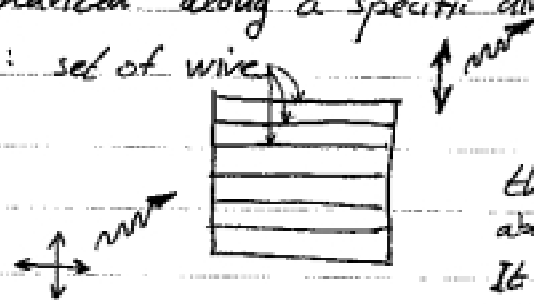
For every propagation direction $\vec{k}$ there are 2 independent linear polarization directions

from which we can build all others.

Most natural ~~sources~~ and artificial light sources (like the neon light ~~we~~ and the sun light in the room) emit unpolarized light that is it contains many waves each polarized at a different direction.

It is possible to pick waves with particular polarization direction by using a <u>Linear Polarizer</u>. They are usually made of long synthetic molecules - polymers. They transmit light only if it is polarized along a specific direction

<u>Another example</u>: set of wires



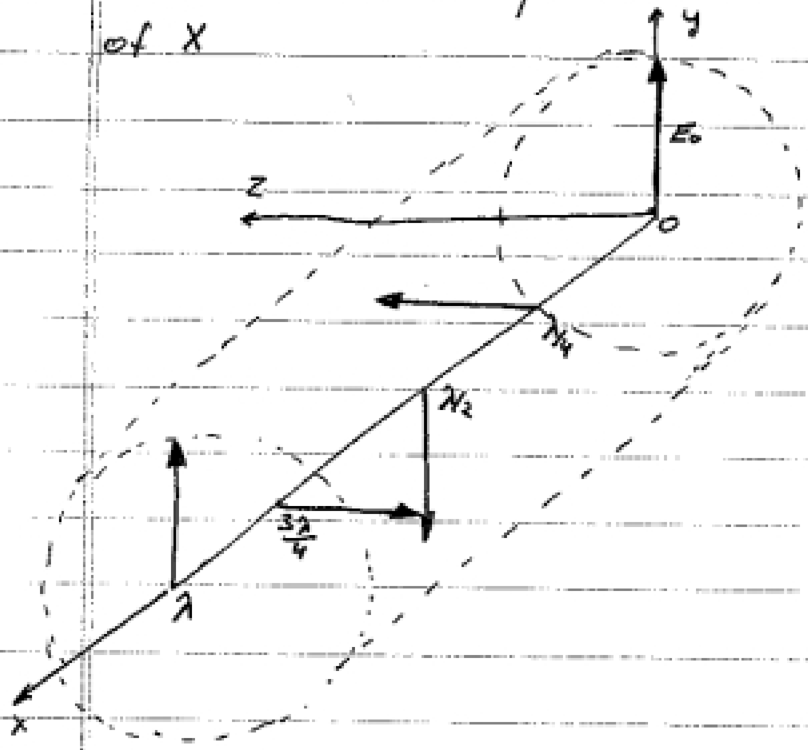
the $\leftrightarrow$ component is absorbed by the wires. It induces currents in them which then produce a field that cancells the $\leftrightarrow$ component.

Another type of polarization is the <u>Circular Polarization</u>.
Consider the following combination of linearly polarized waves with $\frac{\pi}{2}$ phase shift between them:

$$\vec{E} = \frac{E_0}{\sqrt{2}}\left[\sin\left(kx-\omega t+\frac{\pi}{2}\right)\hat{y} + \sin(kx-\omega t)\,\hat{z}\right]$$

$$= \frac{E_0}{\sqrt{2}}\left[\cos(kx-\omega t)\,\hat{y} + \sin(kx-\omega t)\,\hat{z}\right] \xrightarrow[\text{notation}]{\text{in complex}} E_0\left(\frac{\hat{y}-i\hat{z}}{\sqrt{2}}\right)e^{i(kx-\omega t)}$$

Let's look at the polarization direction at $t=0$ as a function of X

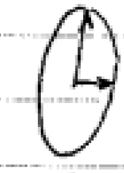


The electric field is tracing a spiral in space

as time goes on this spiral rotates in an anticlockwise sense.

By taking $E_0\left(\frac{\hat{y}+i\hat{z}}{\sqrt{2}}\right)e^{i(kx-\omega t)}$ we get a similar spiral but now it twist in a clockwise direction.

If the two linear components have different amplitudes one obtains the most general case: <u>Elliptical Polarization</u> since a peak in one direction will not equal the other



* Demonstration with Linear Polarizers. (Relation to Stern Gerlach)