Lecture 13

Reading Assignment	Polarization: F I Ch. 33, W 81, 82
	EM Energy: F II Ch. 27

The great thing about Maxwell's contribution is that now the equations

1. $\vec{\nabla}\cdot\vec{E} = 4\pi\rho$ 2. $\vec{\nabla}\cdot\vec{B} = 0$

3. $\vec{\nabla}\times\vec{E} = -\frac{1}{c}\frac{\partial\vec{B}}{\partial t}$ 4. $\vec{\nabla}\times\vec{B} = \frac{4\pi}{c}\vec{J} + \frac{1}{c}\frac{\partial\vec{E}}{\partial t}$

contain within themselves a feed back mechanism between the changes in the electric and magnetic fields. From the third equation we see that

changes in time of $\vec{B}$ $\Longrightarrow$ induce a time dependent $\vec{E}$

but then the fourth equation gives

time dependent $\vec{E}$ $\Longrightarrow$ change in $\vec{B}$

and the cycle closes on itself. The result is electromagnetic waves. We will now show in detail how they arise as a consequence of Maxwell's equations. We can do so using the above form of the equations, it will be, however, easier if we cast them in a new equivalent form first.

We start by noting that the equation $\vec{\nabla}\cdot\vec{B} = 0$ implies that $\vec{B}$ can be written as a curl of some other vector field. This field is called the <u>Vector Potential</u> and is denoted by $\vec{A}$.

Thus we have $\vec{B} = \vec{\nabla}\times\vec{A}$

Plugging this expression into eq. 3 we find

$$\vec{\nabla}\times\vec{E} = \frac{-1}{c}\frac{\partial}{\partial t}\left(\vec{\nabla}\times\vec{A}\right) = -\frac{1}{c}\vec{\nabla}\times\left(\frac{\partial\vec{A}}{\partial t}\right)$$

we again use the fact that we can exchange the order of the time derivative and the space derivatives in ∇.

$$\Rightarrow \vec{\nabla}\times\left(\vec{E}+\frac{1}{c}\frac{\partial\vec{A}}{\partial t}\right) = 0$$

But we know (I told you that without a proof) that if the curl of a vector field vanishes the ~~a~~ vector field can be written as the gradiant of some scalar field. In our case this scalar field is called the <u>Scalar Potential</u> and is denoted by ϕ. So we have

for technical convenience we equate it to grad($-\phi$) rather than grad(ϕ).

$$\vec{E}+\frac{1}{c}\frac{\partial\vec{A}}{\partial t} = -\vec{\nabla}\phi$$

So we can express the electric and magnetic fields through the scalar and vector potentials according to:

$$\vec{E} = -\vec{\nabla}\phi - \frac{1}{c}\frac{\partial\vec{A}}{\partial t}\,, \qquad \vec{B} = \vec{\nabla}\times\vec{A}$$

This way of writing $\vec{E}$ and $\vec{B}$ is not unique and entails a certain freedom which I will now show you.

Consider different scalar and vector potentials ϕ', $\vec{A}'$ which are obtained by adding to $\vec{A}$ a gradient of some function Λ and subtracting $\frac{1}{c}\frac{\partial \Lambda}{\partial t}$ from ϕ:

$$\vec{A}' = \vec{A} + \vec{\nabla}\Lambda \quad ; \quad \phi' = \phi - \frac{1}{c}\frac{\partial \Lambda}{\partial t} \qquad (*)$$

What are the fields associated with $\vec{A}'$ and ϕ'? Let's find out:

$$\vec{B}' = \nabla \times \vec{A}' = \vec{\nabla} \times (\vec{A} + \vec{\nabla}\Lambda) = \vec{\nabla} \times \vec{A} + \overbrace{\vec{\nabla} \times \vec{\nabla}\Lambda}^{=0 \text{ as you showed in your HW.}} = \vec{\nabla} \times \vec{A} = \vec{B}$$

$$\vec{E}' = -\vec{\nabla}\phi' - \frac{1}{c}\frac{\partial \vec{A}'}{\partial t} = -\vec{\nabla}\left(\phi - \frac{1}{c}\frac{\partial \Lambda}{\partial t}\right) - \frac{1}{c}\frac{\partial}{\partial t}\left(\vec{A} + \vec{\nabla}\Lambda\right)$$

$$= -\vec{\nabla}\phi + \underbrace{\frac{1}{c}\vec{\nabla}\frac{\partial \Lambda}{\partial t} - \frac{1}{c}\frac{\partial}{\partial t}\vec{\nabla}\Lambda}_{\substack{\text{cancell each other} \\ \text{since } \frac{\partial}{\partial t}\vec{\nabla} = \vec{\nabla}\frac{\partial}{\partial t}}} - \frac{1}{c}\frac{\partial \vec{A}}{\partial t} = \vec{E}$$

The Result: The fields $\vec{E}, \vec{B}$ DO NOT CHANGE!
We thus have the freedom to change $\vec{A}$ and ϕ according to $(*)$ without effecting $\vec{E}$ and $\vec{B}$ and hence the physics.
This freedom is called <u>Gauge Freedom</u>
Transformations of the type $(*)$ are called <u>Gauge Transformations</u>
Picking one particular $\vec{A}$ and ϕ to describe the fields is called <u>Fixing the Gauge</u>.

By writing $\vec{E} = -\vec{\nabla}\phi - \frac{1}{c}\frac{\partial \vec{A}}{\partial t}$ and $\vec{B} = \nabla \times \vec{A}$

we automatically solved Maxwell's equations # 2 and 3 (we used these eqs. to find the above expressions).
Let us see now what are the other two equations in terms of ϕ and $\vec{A}$:

Eq 1: $\vec{\nabla}\cdot\left(-\vec{\nabla}\phi - \frac{1}{c}\frac{\partial \vec{A}}{\partial t}\right) = 4\pi\rho$

What is $\vec{\nabla}\cdot(\vec{\nabla}\phi)$? $= \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)\cdot\left(\frac{\partial\phi}{\partial x}, \frac{\partial\phi}{\partial y}, \frac{\partial\phi}{\partial z}\right)$

$= \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2} \equiv \nabla^2\phi$: The Laplacian of ϕ.

$\Rightarrow \quad -\nabla^2\phi - \frac{1}{c}\frac{\partial}{\partial t}\vec{\nabla}\cdot\vec{A} = 4\pi\rho$

Eq 4: $\nabla\times(\vec{\nabla}\times\vec{A}) = \frac{4\pi}{c}\vec{J} + \frac{1}{c}\frac{\partial}{\partial t}\left(-\vec{\nabla}\phi - \frac{1}{c}\frac{\partial\vec{A}}{\partial t}\right)$

We now use the identity which you will prove in your HW that: $\vec{\nabla}\times(\vec{\nabla}\times\vec{A}) = \vec{\nabla}(\vec{\nabla}\cdot\vec{A}) - \nabla^2\vec{A}$

$\Rightarrow \quad -\nabla^2\vec{A} + \frac{1}{c^2}\frac{\partial^2\vec{A}}{\partial t^2} + \vec{\nabla}\left(\vec{\nabla}\cdot\vec{A} + \frac{1}{c}\frac{\partial\phi}{\partial t}\right) = \frac{4\pi}{c}\vec{J}$

These equations are written for arbitrary ϕ and $\vec{A}$ and they look complicated. However, we can use the gauge freedom to make them simpler. To this end we fix the gauge by picking ϕ and $\vec{A}$ that obey:

$$\vec{\nabla}\cdot\vec{A} = -\frac{1}{c}\frac{\partial\phi}{\partial t} \quad : \quad \underline{\text{Lorentz Condition}}$$

[If the original $\vec{A}$ and ϕ don't obey this condition we can make a gauge transformation with Λ that obeys the equation

$$\nabla^2\Lambda - \frac{1}{c^2}\frac{\partial^2\Lambda}{\partial t^2} = -\vec{\nabla}\cdot\vec{A} - \frac{1}{c}\frac{\partial\Lambda}{\partial t}$$

$$\Rightarrow \vec{\nabla}\cdot\vec{A}' = \vec{\nabla}\cdot(\vec{A}+\vec{\nabla}\Lambda) = -\frac{1}{c}\frac{\partial}{\partial t}\left(\phi - \frac{1}{c}\frac{\partial\Lambda}{\partial t}\right) = -\frac{1}{c}\frac{\partial\phi'}{\partial t}$$]

In this gauge we find that Eqs 1 and 4 reduce to:

$$\boxed{\begin{aligned} \nabla^2\phi - \frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2} &= -4\pi\rho \\ \nabla^2\vec{A} - \frac{1}{c^2}\frac{\partial^2\vec{A}}{\partial t^2} &= -\frac{4\pi}{c}\vec{J} \end{aligned}}$$

These 4 equations together with Lorentz condition are <u>equivalent</u> to to the original set of Maxwell's equations. They have the appearance of <u>wave equations</u> for ϕ, A_x, A_y, A_z with ρ, J_x, J_y, J_z appearing as source terms on the r.h.s. Next time we will solve them.

[By the way, the Lorentz condition does not specify $\vec{A}$ and ϕ completely. You may easily check that if $\vec{A}$ and ϕ obey the condition so do $\vec{A}' = \vec{A}+\vec{\nabla}\Lambda$ and $\phi' = \phi - \frac{1}{c}\frac{\partial\Lambda}{\partial t}$ as long as $\nabla^2\Lambda - \frac{1}{c^2}\frac{\partial^2\Lambda}{\partial t^2} = 0$: Another wave equation !!!]