Lecture 12

The entire field of electromagnetism can be summarized in 4 equations - the Maxwell's equations. They describe for us how the electric and magnetic fields are generated by charges and currents, and how these fields evolve in time. Together with Lorentz force law $\vec{F} = q(\vec{E} + \frac{\vec{J}\times\vec{B}}{c})$ and Newton's second law of motion $\vec{F} = \frac{d\vec{P}}{dt}$ they provide a complete description of the coupled system of the electromagnetic field and charges.

We will briefly go over the 4 equations, write them in their differential and integral forms and comment on their physical meaning.

Note I'm using the CGS system of units in which Coulomb's law read $\vec{F} = \frac{q_1 q_2}{r_{12}^2}\hat{r}_{12}$. Feynman (and others) are using the SI system where

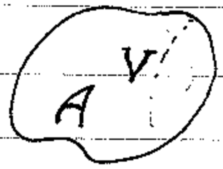
$$\vec{F} = \frac{1}{4\pi\epsilon_0}\frac{q_1 q_2}{r_{12}^2}\hat{r}_{12}$$

Be aware of this difference when you compare my notes with Feynman's book.

In the CGS system the electric and magnetic fields are measured in the same units. This is in accord with the fact that they are different aspects of the same underlying entity.

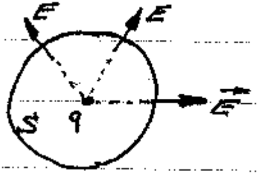
Differential Form   Integral Form

1. $\vec{\nabla}\cdot\vec{E} = 4\pi \underbrace{\rho}_{\text{charge density}} \xrightarrow{\text{Gauss Law}} \int_A \vec{E}\cdot d\vec{a} = 4\pi\int_V \rho\, dv = \underbrace{4\pi Q}_{\text{total charge inside V}}$



Meaning: The flux of the electric field through a closed surface = $4\pi \times$ Charge inside the volume defined by the surface.

This is nothing but Coulomb's law: Consider a point charge q at the center of a sphere of radius R. The electric field produced by the charge has a value of $\frac{q}{R^2}$ on the sphere and is perpendicular to the surface. $\Rightarrow$ at each point $\vec{E}\cdot d\vec{a} = E\,da = \frac{q}{R^2}da$ summing over the area elements



$$\int_S \vec{E}\,d\vec{a} = \frac{q}{R^2}\cdot \text{surface area of } S = \frac{q}{R^2}\,4\pi R^2 = 4\pi q$$

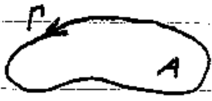
2. $\vec{\nabla}\cdot\vec{B} = 0 \xrightarrow{\text{Gauss Law}} \int_A \vec{B}\cdot d\vec{a} = 0$

Meaning: The flux of $\vec{B}$ through any closed surface vanishes. There are NO magnetic charges (monopols)

→ none were observed so far but they are pedicted to exist by some theories.

Differential Form   Integral Form

3. $\vec\nabla\times\vec E = -\frac{1}{c}\frac{\partial\vec B}{\partial t} \xrightarrow{\substack{\text{Stokes'}\\ \text{Theorem}}} \oint_\Gamma \vec E\cdot d\vec l = -\frac{1}{c}\frac{\partial}{\partial t}\int_A \vec B\cdot d\vec a$



Meaning: Faraday's law of induction :
The circulation of $\vec E$ around a loop = $-\frac{1}{c}\frac{\partial}{\partial t}$(Flux of $\vec B$ through the loop).

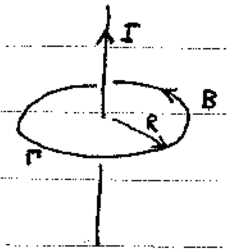
Before Maxwell it was believed that the fourth equation is

4. (wrong) $\vec\nabla\times\vec B = \frac{4\pi}{c}\vec J \xrightarrow{\substack{\text{Stokes'}\\ \text{Theorem}}} \oint_\Gamma \vec B d\vec l = \frac{4\pi}{c}\int_A \vec J\cdot d\vec a = \frac{4\pi}{c} I$

current density   total current through A

Meaning: This is just Amper's or Biot-Savart's laws
The circulation of $\vec B$ around a loop = $\frac{4\pi}{c}$ × current through the loop.

Consider for example a current carrying wire :
Amper's law tells us that the magnetic field generated by the current is directed in the azimuthal direction and has a magnitude of $B = \frac{2I}{c}\frac{1}{R}$



⇒ at each point on the circle $\vec B\cdot d\vec l = B\,dl = \frac{2I}{c}\frac{1}{R}$

⇒ $\int_\Gamma \vec B d\vec l = \frac{2I}{c}\frac{1}{R}\cdot$ length of Γ $= \frac{2I}{c}\frac{1}{R}\cdot 2\pi R = \frac{4\pi}{c} I$

What is wrong with 4.(wrong) ?

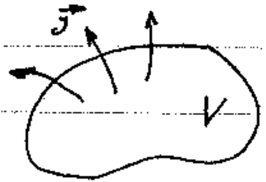
Take the divergence of both sides of the equation. You will show in your homework that the divergence of a curl of a vector field vanishes ⇒ $\vec{\nabla}\cdot\vec{\nabla}\times\vec{B} = 0$ so the equation tells us

$$\vec{\nabla}\cdot\vec{J} \overset{?}{=} 0$$

is this always true ?

Answer: No. Consider the integral of $\vec{\nabla}\cdot\vec{J}$ over a volume V

$$\int_V \vec{\nabla}\cdot\vec{J}\, d\tau \overset{\substack{\text{Gauss}\\ \text{theorem}}}{=} \int_A \vec{J}\cdot d\vec{a} = \underset{\text{through } A}{\text{flux of } \vec{J}}$$



But we showed that the flux of $\vec{J}$ through a surface A is the number of charges that cross A in unit time! So:

$$\int_V \vec{\nabla}\cdot\vec{J}\, d\tau = \underset{\text{through } A}{\text{Flux of } \vec{J}} = \underset{\uparrow}{-}\frac{\partial}{\partial t}\left[\text{total charge in } V\right] = -\frac{\partial}{\partial t}\int_V \rho\, d\tau$$

if the flux out of
V is positive it means
that the charge inside
V decreases in time

⇒ If the number of charges is conserved, if no charges are created or destroyed we must have:

$\vec{\nabla}\cdot\vec{J} = -\frac{\partial \rho}{\partial t}$ The Continuity Equation

Since all of our observations so far tell us that the charge in the universe is conserved we must assume the continuity equation.

We thus see that 4. (wrong) is true only when $\vec{\nabla}\cdot\vec{J} = 0 \Leftrightarrow \frac{\partial \rho}{\partial t} = 0$
This is the case when the currents do not change in time: Magnetostatics.

We want to correct eq. 4 so it will be consistent with the continuity equation. To do so we must add to it a piece whose divergence will give us the missing $\frac{\partial \rho}{\partial t}$. The answer is:

4. (correct):

$$\vec{\nabla}\times\vec{B} = \frac{4\pi}{c}\vec{J} + \overset{\text{Maxwell's great contribution}}{\boxed{\frac{1}{c}\frac{\partial \vec{E}}{\partial t}}} \xrightarrow{\text{Stokes' theorem}} \int_{\Gamma} \vec{B}\, d\vec{l} = \frac{4\pi}{c}\int_{A} \vec{J}\, d\vec{a} + \frac{1}{c}\frac{\partial}{\partial t}\int_{A} \vec{E}\cdot d\vec{a}$$

Check that now it works: take again the divergence ($\vec{\nabla}\cdot\vec{\nabla}\times B = 0$)

$$0 = \frac{4\pi}{c}\vec{\nabla}\cdot\vec{J} + \frac{1}{c}\vec{\nabla}\cdot\frac{\partial \vec{E}}{\partial t} \underset{\substack{\uparrow \\ \text{we can exchange the order of} \\ \text{the space der. in } \vec{\nabla} \text{ and the} \\ \text{time derivative.}}}{=} \frac{4\pi}{c}\vec{\nabla}\cdot\vec{J} + \frac{1}{c}\frac{\partial}{\partial t}\vec{\nabla}\cdot\vec{E} \underset{\substack{\uparrow \\ \text{using Maxwell eq. \#1.}}}{=} \frac{4\pi}{c}\vec{\nabla}\cdot\vec{J} + \frac{4\pi}{c}\frac{\partial \rho}{\partial t}$$

$$\Rightarrow \vec{\nabla}\cdot\vec{J} = -\frac{\partial \rho}{\partial t}$$

Meaning: The circulation of $\vec{B}$ around a loop $= \frac{4\pi}{c} \times$ current through loop
$+\frac{1}{c}\frac{\partial}{\partial t}$(Flux of $\vec{E}$ through loop)