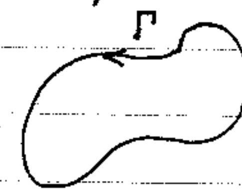
Lecture 11   Reading assignment : EM waves   F II Chap. 20

The circulation is defined for a closed curve



Note: The curve has direction. The convention is that the positive direction is counter clockwise.

The Curl of a vector field $\vec{V}$ at a point $\vec{r}$ is a vector. It is defined as following:

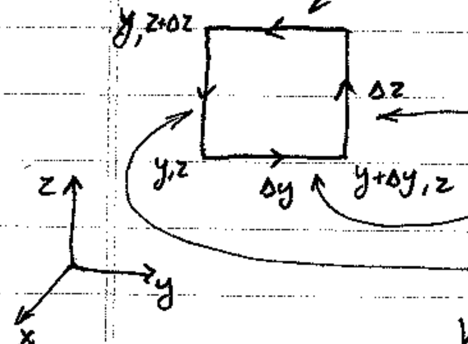
Consider a small area element Δa around $\vec{r}$.

As we learned the vector $\vec{\Delta a}$ has length Δa and is directed along the normal to the element. Then:

The component of the vector (Curl $\vec{V}$) along the direction of $\vec{\Delta a}$

$$\equiv \frac{\text{circulation of } \vec{V} \text{ around } \Gamma}{\Delta a}$$

Let us calculate for example the X component of Curl $\vec{V}$:
For this we have to consider a small area element with a normal in the x direction. i.e. the element is parallel to the y-z plane



$$\left(\text{Curl } \vec{V}\right)_x = \frac{V_y(x,y,z)\,\Delta y - V_y(x,y,z+\Delta z)\,\Delta y}{\Delta y\,\Delta z} + \frac{V_z(x,y+\Delta y,z)\,\Delta z - V_z(x,y,z)\,\Delta z}{\Delta y\,\Delta z}$$

$$= \frac{V_y(x,y,z) - V_y(x,y,z+\Delta z)}{\Delta z} + \frac{V_z(x,y+\Delta y,z) - V_z(x,y,z)}{\Delta y} \xrightarrow{\Delta y,\Delta z \to 0} \frac{\partial V_z}{\partial y} - \frac{\partial V_y}{\partial z}$$

You can convince yourself that the other component of curl $\vec{V}$ are

$$(\text{curl}\,\vec{V})_y = \frac{\partial V_x}{\partial z} - \frac{\partial V_z}{\partial x}$$

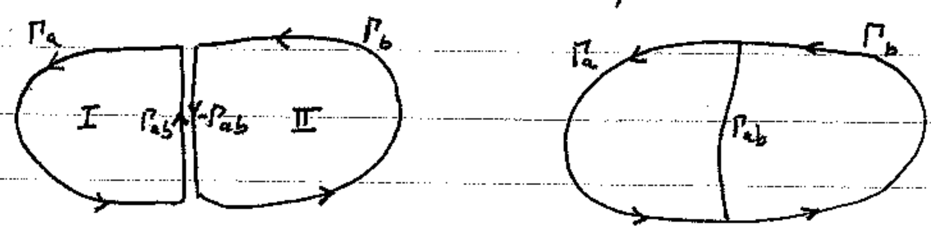
$$(\text{curl}\,\vec{V})_z = \frac{\partial V_y}{\partial x} - \frac{\partial V_x}{\partial y}$$

$\Rightarrow$ We can write the vector curl $\vec{V}$ as: $\text{curl}\,\vec{V} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ V_x & V_y & V_z \end{vmatrix} = \vec{\nabla} \times \vec{V}$

The Curl produces a vector field out of a vector field

cross product between the vector operator $\vec{\nabla}$ and the vector $\vec{V}$.

Stokes Theorem: Consider an area enclosed by a curve Γ and now divide it into two parts



$$\text{Circulation around I} = \int_{\Gamma_a} \vec{V} d\vec{l} + \int_{\Gamma_{ab}} \vec{V} d\vec{l}$$

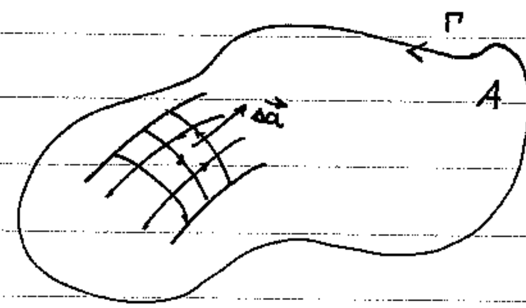
$+$

$$\text{circulation around II} = \int_{\Gamma_b} \vec{V} d\vec{l} - \int_{\Gamma_{ab}} \vec{V} \cdot d\vec{l}$$

we traverse Γ_{ab} in the opposite direction

$\Rightarrow$ circulation around I + circulation around II $= \int_{\Gamma_a} \vec{V} d\vec{l} + \int_{\Gamma_b} \vec{V} d\vec{l} =$ circulation around Γ.

$\Rightarrow$ Now consider an area A enclosed by Γ and devide it into many infinitesimal element $\Delta\vec{a}$:



Have that:

$$\begin{array}{ccc} \text{Circulation around } \Gamma & = & \sum \text{circulations around the elements} \\ \| & & \| \quad \text{: from def of } \vec{\nabla}\times\vec{V} \\ \int_\Gamma \vec{V}\cdot d\vec{l} & & \sum \left(\text{curl } \vec{V} \text{ at the center of element } \Delta a\right)\cdot \Delta\vec{a} \\ & & \Big\downarrow{\scriptstyle \Delta a \to 0} \\ & & \int_A \text{curl}\vec{V}\cdot d\vec{a} \end{array}$$