Lecture 10 Reading assignment : Maxwell's equations FII ch 18.

Since we are going to generalize our discussion to waves in dimension greater than 1 and in particular study EM waves using Maxwell's equation it's important you that you'll be familiar with the basic notions of vector fields and the differential operations involving them.

Basic Definitions :

Scalar Field : A field which is characterized at each point by a single number – a scalar (the number may change in time, but we don't worry about that for the moment)

Examples: 1. Temperature field $T(x,y,z)$

2. Density field – what is the density of the air, say, at each point in space $\rho(x,y,z)$

Vector Field : Characterized by a vector at each point

Examples: 1. Velocity field of a fluid gives the velocity (magnitude and direction of the fluid particles) $\vec{v}(x,y,z)$

2. Electric field, Magnetic field $\vec{E}(x,y,z)$, $\vec{B}(x,y,z)$.

The Gradient Consider a scalar field , let's say the temperature, at a point $\vec{r}=(x,y,z)$ and ask by how much does it change if we move to a neighboring point $\vec{r}+\Delta\vec{r}$; $\Delta\vec{r}=(\Delta x, \Delta y, \Delta z)$
The aswer is of course

$$\Delta T = T(\vec{r}+\Delta\vec{r}) - T(\vec{r}) = \frac{\partial T}{\partial x}\cdot\Delta x + \frac{\partial T}{\partial y}\cdot\Delta y + \frac{\partial T}{\partial z}\Delta z$$

where the partial derivatives are evaluated at the point $\vec{r}$.
We now define a new vector $\vec{\nabla}T$. It is called the <u>Gradient</u> of T and its components are

$$\vec{\nabla}T = \left(\frac{\partial T}{\partial x}, \frac{\partial T}{\partial y}, \frac{\partial T}{\partial z}\right)$$

We can then write the change in T as a scalar product of the vectors $\vec{\nabla}T$ and $\Delta\vec{r}$: $\Delta T = \vec{\nabla}T \cdot \Delta\vec{r}$

It is also ~~customary~~ useful to introduce the <u>Vector operator</u> <u>Nabla</u> defined as:

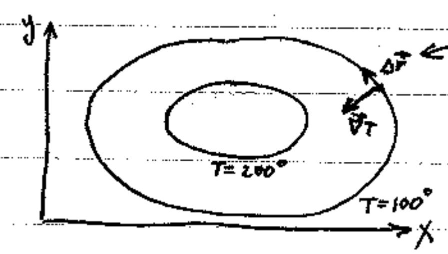
$$\vec{\nabla} = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)$$

The gradient is then the result of operating with $\vec{\nabla}$ on a scalar function, in our example T.

a scalar field

<u>Note:</u> $\vec{\nabla}T$ = grad T = vector field

<u>Example</u> Temperature field of a metalic plate



Consider the infinitesimal vector $\Delta\vec{r}$ connecting two neighboring points on an equi temperature curve. Since $\Delta T = \vec{\nabla}T \cdot \Delta\vec{r} = 0$ between the two points We conclude that

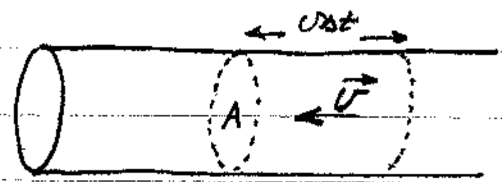
a topographic map of the function T(X,y)

The vector $\vec{\nabla}T$ is <u>perpendicular</u> to lines (surfaces) of constant T.

It points in the <u>direction of largest change</u> in T.

Flux of a vector field

Consider a fluid flowing down a tube. We may ask the question what is the number of fluid particles that ~~cross~~ flow through a crossection of area A which is perpendicular to the flow per unit time

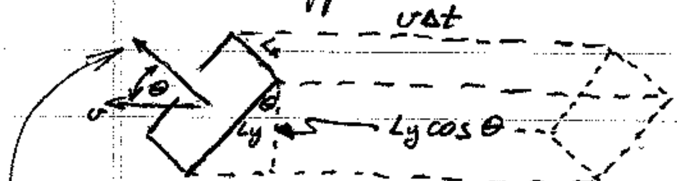


It is clear that during the time Δt all the particles that are at a distance $v\Delta t$ or less from A will cross it.

$$\Rightarrow \frac{\#\text{ of particles crossing } A \text{ in } \Delta t}{\Delta t} = \frac{\underbrace{A \cdot v\Delta t}_{\substack{\text{volume of cylinder}\\ \text{containing the particles that}\\ \text{will cross in } \Delta t}} \cdot \underbrace{\text{density of particles}}_{\rho = \#\text{ of particles in unit volume}}}{\Delta t}$$

$$= \rho v \cdot A$$

What happens if the area A is tilted with respect to v?



The Normal to A

The volume containing the particles that cross A in Δt is now

$$L_x \cdot L_y \cos\theta \cdot v\Delta t = A v \Delta t \cdot \cos\theta$$

If we now define a <u>vector</u> $\vec{A}$ with magnitude A and direction perpendicular to A, and the vector field $\vec{J} = \rho\vec{v}$ we can write

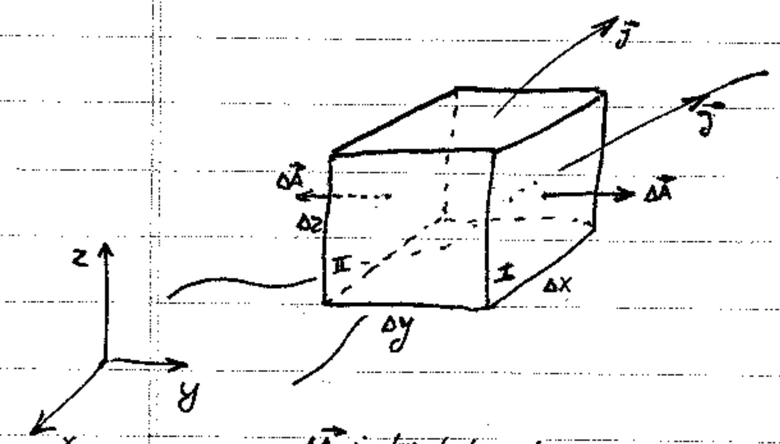
$$\frac{\#\text{ of particles crossing } A \text{ in } \Delta t}{\Delta t} = J A\cos\theta = \vec{J}\cdot\vec{A}$$

The vector field $\vec{J}$ is called the <u>current density</u> field. Its direction is along $\vec{v}$ and its magnitude is the number of particles crossing a unit area perpendicular to $\vec{v}$ in unit time.

$\vec{J}\cdot\vec{A}$ is the <u>Flux</u> of the vector field $\vec{J}$ through the area A.

Note that the vector $\vec{A}$ contains information both on the size of the area A and its orientation.

Let us now consider a small rectangular volume element $\underbrace{\Delta x\Delta y\Delta z}_{=\Delta V}$ and ask what is the flux of $\vec{J}$ through its ~~boundary~~ surface:



The flux through side I is then
$J_y(x, y+\Delta y, z)\cdot\Delta x\Delta z$

The flux through side II is

$\Delta\vec{A}$ is directed out of the element, i.e. in the negative y direction. → $-J_y(x, y, z)\cdot\Delta x\Delta z$

We thus find using similar considerations for the other sides:

$$\frac{\text{Total flux of } \vec{J} \text{ out of } \Delta V}{\Delta V} = \frac{J_x(x+\Delta x, y, z) - J_x(x,y,z)}{\Delta x}$$

$$+ \frac{J_y(x, y+\Delta y, z) - J_y(x,y,z)}{\Delta y}$$

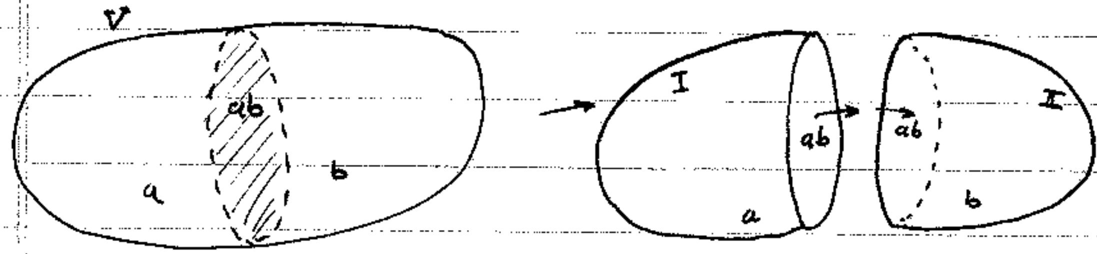
$$+ \frac{J_z(x,y,z+\Delta z) - J_z(x,y,z)}{\Delta z}$$

$$\xrightarrow{\Delta x, \Delta y, \Delta z \to 0} \frac{\partial J_x}{\partial x} + \frac{\partial J_y}{\partial y} + \frac{\partial J_z}{\partial z}$$: The Divergence of $\vec{J}$

Can write as : $= \vec{\nabla} \cdot \vec{J}$

Gives a scalar field from a vector field.

Gauss Theorem: Consider a volume V and separate it into two sub volumes I and II



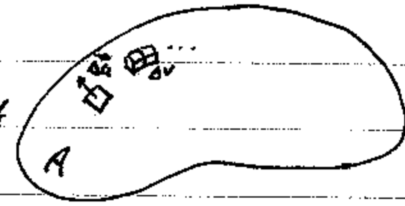
The flux of a vector field from volume I = flux through a + flux through ab

" II = flux through b − flux through ab

⇒ Flux from I + Flux from II = flux through a + flux through b = Flux from V.

The point to notice : The fluxes on the common side cancel each other.

Now devide V into many sub volumes ΔV
It will result in the surface A devided into element Δa



Flux of $\vec{J}$ through A $= \sum_i \vec{J}_i \cdot \Delta\vec{a}_i$

$|||$

Flux of $\vec{J}$ from V

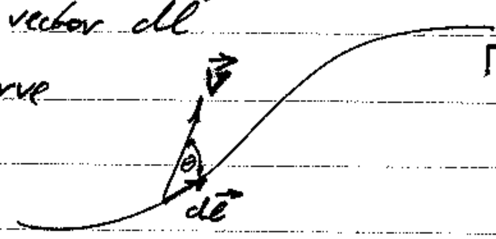
$||$

$\sum_j$ flux from ΔV_j $= \sum_j$ divergence of J at the place of $\Delta V_j \cdot \Delta V$

$$\longrightarrow \int_A \vec{J}\, d\vec{a} = \int_V \vec{\nabla}\cdot\vec{J}\, dv$$

The Circulation of a vector field $\vec{V}$ around a closed curve Γ is the line integral of $\vec{V}$ along the curve:

Devide the curve into small elements of length dl
assign to each of them a small vector $d\vec{l}$
of length dl, tangent to the curve



The line integral of $\vec{V}$ along Γ
is then defined as

$$\int_\Gamma \vec{V}\cdot d\vec{l}$$

: it's a sum of the tangent component of $\vec{V}$ times the line element dl

$$\vec{V}\cdot d\vec{l} = \overbrace{V\cdot\cos\theta}\cdot dl$$