Homework Assignment #3 **Physics 17 Fall 99**

Due: Thursday, October 28

1. The phase velocity v_φ of deep water waves is given by

$$v_\varphi = \sqrt{\frac{g}{k}}\,, \tag{1}$$

where g is the acceleration due to gravity and k is the wave number.

   (a) Find the group velocity of the waves.

   (b) Deduce the wave equation for the displacement ψ associated with these waves.

2. The dispersion relation for an electromagnetic wave propagating through a plasma is

$$\omega = \sqrt{\omega_p^2 + c^2k^2}\,, \tag{2}$$

where ω_p is a constant called the plasma frequency and c is the velocity of light.

   (a) Calculate and plot the phase and group velocities as a function of k.

   (b) Calculate the product $v_\varphi v_g$.

3. Consider a string that at $t = 0$ has the shape of a localized pulse which is propagating to the right and is of the form

$$\psi(x, t = 0) = e^{-\frac{x^2}{2l^2}}\,, \tag{3}$$

where l is a constant (how is it related to the width of the pulse?). Such a localized pulse (not necessarily of that particular functional form) is called a wave packet.

   (a) Calculate the Fourier transform $\tilde{\psi}(k, t = 0)$ of $\psi(x, t = 0)$. You will find the following integral

$$\int_{-\infty}^{\infty} dx\, e^{-\alpha x^2} = \sqrt{\frac{\pi}{\alpha}} \quad \text{for } \mathrm{Re}(\alpha) > 0 \tag{4}$$

   useful.

   (b) Show that the width Δx of the wave packet and the width Δk of $\tilde{\psi}(k)$ in k space obey $\Delta x \Delta k = 1$. This is an example of the "uncertainty relation". Broad features in x space are narrow in k space and vice versa.

(c) Calculate $\psi(x,t)$ at latter times assuming that the string is described by a linear wave equation with dispersion law $\omega = vk$.

(d) Repeat (3c) but now assuming $\omega = vk + \alpha k^2$. What happens to the pulse in this case?

4. Consider a spherically symmetric scalar field, that is, a field which depends only on the distance r from the origin: $\phi(\sqrt{x^2+y^2+z^2})$.

   (a) Calculate $\nabla\phi(r)$.

   (b) For the specific case $\phi(x,y,z) = (x^2+y^2+z^2)^{-\frac{3}{2}}$ calculate the magnitude and direction of $\nabla\phi$ at the point $(1,2,3)$.

5. (a) Show that if a vector field $\mathbf{v}(x,y,z)$ can be written as a gradient of a scalar field $\phi(x,y,z)$ its curl vanishes, i.e.

$$\nabla \times \nabla\phi = 0\,. \tag{5}$$

   (b) Show that if a vector field $\mathbf{v}(x,y,z)$ can be written as a curl of another vector field field $\mathbf{u}(x,y,z)$ its divergence vanishes, i.e.

$$\nabla \cdot \nabla \times \mathbf{u} = 0\,. \tag{6}$$

6. (a) Show that for a vector field $\mathbf{v}$ and a scalar field ϕ

$$\nabla \cdot (\phi\mathbf{v}) = (\nabla\phi)\cdot\mathbf{v} + \phi(\nabla\cdot\mathbf{v})\,. \tag{7}$$

   (b) Use the result of (6a) to prove the identity between the following two expressions for the energy of an electrostatic field

$$\frac{1}{2}\int d^3x\,\phi\rho = \frac{1}{8\pi}\int d^3x\,\mathbf{E}^2\,, \tag{8}$$

   where $\mathbf{E} = -\nabla\phi$. Assume that the electrostatic field vanishes at infinity.

7. (a) Consider the vector field

$$\mathbf{F} = -\frac{y}{x^2+y^2}\mathbf{i} + \frac{x}{x^2+y^2}\mathbf{j}\,, \tag{9}$$

   where $\mathbf{i}$ and $\mathbf{j}$ are unit vectors along the x and y directions respectively. Calculate the circulation of $\mathbf{F}$ around the unit circle in the xy plane.

   (b) Consider now the vector field $\mathbf{F} = -y\mathbf{i} + x\mathbf{j}$. Show, using Stokes theorem, that the circulation of $\mathbf{F}$ around any closed curve Γ in the xy plane is given by

$$\oint_\Gamma \mathbf{F}\cdot d\mathbf{l} = 2A\,, \tag{10}$$

   where A is the area enclosed by Γ.