Note Title 7/8/2018

1a. $C_j = \frac{1}{\sqrt{N}} \sum_k C_k e^{ikj}$ $k = \frac{2\pi}{N} \cdot n$ $n = -\frac{N}{2}+1, \cdots, \frac{N}{2}$: assume even N.

$$H = \frac{1}{N} \sum_{kk'} \sum_{j=1}^{N} \Big\{ -t\left[C_k^+ C_{k'} e^{-ik(j+1)+ik'j} + H.c \right] - \mu C_k^+ C_{k'} e^{i(k'-k)j}$$

$$-\Delta \left[C_k^+ C_{k'}^+ e^{-ik(j+1)-ik'j} + H.c \right] \Big\}$$

$$= \sum_k \overbrace{(-2t\cos k - \mu)}^{\xi_k} C_k^+ C_k - \Delta \sum_{k=-\frac{N}{2}+1}^{\frac{N}{2}-1} \left(e^{-ik} C_k^+ C_{-k}^+ + e^{ik} C_{-k} C_k \right)$$

$$+\Delta \left(\cancel{C_\pi^+ C_\pi^+ + H.c} \right) \xrightarrow{} 0 \text{ due to Fermi statistics}$$

$$= \sum_{\pi > k > 0} \left[\xi_k (C_k^+ C_k + C_{-k}^+ C_{-k}) - (2i\Delta \sin k\, C_k^+ C_{-k}^+ + H.c) \right] + \xi_0 C_0^+ C_0 + \xi_\pi C_\pi^+ C_\pi$$

$$= \sum_{\pi > k > 0} \begin{pmatrix} C_k^+ & C_{-k} \end{pmatrix} \begin{pmatrix} \xi_k & -2i\Delta \sin k \\ 2i\Delta \sin k & -\xi_k \end{pmatrix} \begin{pmatrix} C_k \\ C_{-k}^+ \end{pmatrix} + \sum_{\pi > k > 0} \xi_k + \xi_0 C_0^+ C_0 + \xi_\pi C_\pi^+ C_\pi$$

b. The Hamiltonian has the same structure as for spinfull s-wave BCS Hamiltonian. We can therefore immediately deduce (or by explicitly diagonalizing H for each k)

$$E_k = \sqrt{\xi_k^2 + 4\Delta^2 \sin^2 k} \qquad k = \frac{2\pi}{N}, \cdots \pi - \frac{2\pi}{N}, \quad E_0 = \xi_0, \quad E_\pi = \xi_\pi$$

with each level (except $k=\pi$) doubly degenerate.

c. Because $\Delta(k) = 2i\Delta \sin k$ it changes sign under $k \to -k$ reflecting p-wave symmetry which is to be expected since we deal here with spinless (or equivalently spin polarized) electrons whose spin part of the wave function is symmetric.

d. Since for $\pi > k > 0$ $\xi_k^2 > 0$ it follows that $E_k > 0$ there. E_k vanishes for $k=0$ regardless of Δ if $\mu = -2t$ and for $k=\pi$ regardless of Δ if $\mu = 2t$. It can be shown that the topological phase occurs in the region $|\mu| < 2t$.

2a. From the Kubo formalism we know that

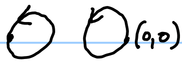
$$\langle \delta n(k,\omega) \rangle = \Pi(k,\omega) \cdot \phi$$

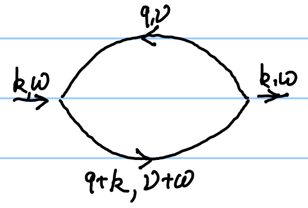
where $\Pi(k,\omega)$ is the (Fourier transformed) $\delta n \cdot \delta n$ retarded correlation function. It can be calculated by analytically continuing the Matsubara correlation function

$$\Pi(k,\omega_n) = -\int_0^\beta d\tau \int dx\, e^{-i(kx-\omega_n\tau)} \langle \delta n(x,\tau)\, \delta n(0,0) \rangle$$

$$= -\int_0^\beta d\tau \int dx\, e^{-i(kx-\omega_n\tau)} \langle (n(x,\tau)-\bar{n})(n(0,0)-\bar{n}) \rangle$$

$$= -\int_0^\beta d\tau \int dx\, e^{-i(kx-\omega_n\tau)} \langle n(x,\tau)\, n(0,0) \rangle - \bar{n}^2$$

The disconnected part $_{x\tau}$  $_{(0,0)}$ is cancelled by $-\bar{n}^2$ and we are left with the connected diagram



$$= (-)(-1)\, \frac{1}{\beta} \sum_{\nu_n} \int \frac{dq}{2\pi}\, \mathcal{G}_0(q, i\nu_n)\, \mathcal{G}_0(q+k, \nu_n+\omega_n)$$

def of Π fermionic loop

Here ω_n is a bosonic Matsubara freq. and ν_n are fermionic freqs.

$$= \frac{1}{\beta} \sum_{i\nu_n} \int \frac{dq}{2\pi} \frac{1}{i\nu_n - \xi_q} \frac{1}{i(\nu_n+\omega_n) - \xi_{q+k}} \qquad \xi_q = \frac{q^2}{2m} - \frac{k_F^2}{2m}$$

$T=0 \Rightarrow \mu = E_F$

$$= \frac{1}{\beta} \sum_{i\nu_n} \int \frac{dq}{2\pi} \frac{1}{i\omega_n + \xi_q - \xi_{q+k}} \left[\frac{1}{i\nu_n - \xi_q} - \frac{1}{i(\nu_n+\omega_n) - \xi_{q+k}} \right]$$

Carrying out the sum over Matsubara frequencies:

$$= \int \frac{dq}{2\pi} \frac{1}{i\omega_n + \xi_q - \xi_{q+k}} \left[n_F(\xi_q) - n_F(\xi_{q+k}) \right]$$

We are at $T=0$ and the square brackets don't vanish only if

$$|q| < k_F \text{ and } |q+k| > k_F \quad \overset{k>0}{\Downarrow} \quad \Rightarrow k_F - k < q < k_F$$

$$|q| > k_F \text{ and } |q+k| < k_F \quad \Rightarrow -k_F - k < q < -k_F$$

$$= \left(\int_{k_F-k}^{k_F} \frac{dq}{2\pi} - \int_{-k_F-k}^{-k_F} \frac{dq}{2\pi} \right) \frac{1}{i\omega_n - \frac{kq}{m} - \frac{k^2}{2m}}$$

Analitically continuing $i\omega_n \to \omega + i\delta$ we find

$$\chi(k,\omega) = \frac{m}{2\pi k} \left\{ \ln \frac{\omega + i\delta + \frac{k_F k}{m} - \frac{k^2}{2m}}{\omega + i\delta + \frac{k_F k}{m} + \frac{k^2}{2m}} - \ln \frac{\omega + i\delta - \frac{k_F k}{m} - \frac{k^2}{2m}}{\omega + i\delta - \frac{k_F k}{m} + \frac{k^2}{2m}} \right\}$$

b. $$\chi(k \to 0, \omega = 0) \to \frac{m}{2\pi k} \left\{ \ln \frac{1 - \frac{k}{2k_F}}{1 + \frac{k}{2k_F}} - \ln \frac{1 + \frac{k}{2k_F}}{1 - \frac{k}{2k_F}} \right\} \to -\frac{m}{\pi k_F}$$

$$= -g(E_F)$$

$$\left(N = \frac{2k}{2\pi / L} = \frac{kL}{\pi} \Rightarrow g(E) = \frac{1}{L} \frac{dN}{dE} = \frac{1}{L} \frac{dN}{dk} \frac{dk}{dE} = \frac{1}{\pi} \frac{m}{k} \right)$$

$$\chi(k = 2k_F + \Delta k, \omega = 0) \sim \frac{m}{2\pi k_F} \ln\left(\Delta k \cdot \frac{k_F}{m} \right)$$

This signals the tendency of the system to spontaneously develop $2k_F$ density modulation (charge density wave). Such a state spontaneously breaks the translational symmetry of H. However due to Mermin-Wagner theorem such instability cannot materialize in 1d in contrast to 2d and 3d.