# Advanced Quantum Mechanics A – Problem Set 6

1. a. Show that for $n$ complex variables $z_i$, $n$ complex parameters $J_i$, and a Hermitian $n \times n$ matrix $H$

$$Z \equiv \int_{-\infty}^{\infty} \prod_{j=1}^{n} \frac{dz_j^* dz_j}{2\pi i} e^{-\sum_{i,j} z_i^* H_{ij} z_j + \sum_i (J_i^* z_i + z_i^* J_i)} = (\det H)^{-1} e^{\sum_{i,j} J_i^* (H^{-1})_{ij} J_j}$$

b. Use this result to calculate the discrete version (in the limit where the number of time slices goes to infinity) of the path integral for a system of free bosons.

c. Show that $\left\langle z_m z_n^* \right\rangle \equiv \frac{1}{Z} \int_{-\infty}^{\infty} \prod_{j=1}^{n} \frac{dz_j^* dz_j}{2\pi i} z_m z_n^* e^{-\sum_{i,j} z_i^* H_{ij} z_j} = H_{mn}^{-1}$

d. Prove Wick's theorem:

$$\left\langle z_{i_1} z_{i_2} \cdots z_{i_N} z_{j_1}^* z_{j_2}^* \cdots z_{j_N}^* \right\rangle = \sum_P H_{i_1 j_{P(1)}}^{-1} \cdots H_{i_N j_{P(N)}}^{-1} = \sum_P \left\langle z_{i_1} z_{j_{P(1)}}^* \right\rangle \cdots \left\langle z_{i_N} z_{j_{P(N)}}^* \right\rangle$$

where the sum is over all permutations of $1, 2, \cdots, N$.

2.a. Show that the Euler-Lagrange equations for a Bose liquid with short-range interactions possess a solution of the form $\phi(r, \theta, z) = e^{in\theta} f(r / r_0)$ where $n \neq 0$ is an integer and $\vec{r} = (r, \theta, z)$. The parameter $r_0$ is the typical length in the problem. What is it? Find the asymptotic behavior of $f(r)$ for small and large $r$.

b. Calculate the current density carried by the solution and show that it describes a vortex around the origin. Calculate the circulation around the origin.

c. Assume that the liquid is inside a cylinder of radius $R$ and calculate the main contribution to the energy of vortex in the limit of large $R$. For which $n$ is this energy the lowest?

d. We can estimate the entropy that accompanies a solution of this form if we divide the cross section of the cylinder into cells of area $r_0^2$ and assume that the vortex core can lie inside any one of them. At what temperature the free-energy of the solution turns negative? At temperatures higher than that vortices would spontaneously appear in the system.

3. Calculate using Bogoliubov model for a Bose liquid with short-range interactions the relative fraction of the liquid that is not in the condensate: $\frac{N - N_0}{N}$.