Advanced Quantum Mechanics A – Problem Set 3

1. a. Show that the second quantized form of the particles' density operator is $\rho(x)=\psi^{\dagger}(x)\psi(x)$.

b. Show that the electronic spin operators S_{μ} $(\mu=x,y,z)$ are given by $S_{\mu}=\frac{1}{2}\sum_{\tau,\tau'=\uparrow,\downarrow}a_{\tau}^{\dagger}(\sigma_{\mu})_{\tau,\tau'}a_{\tau'}$, where a_{τ} annihilates an electron with spin z-component τ and σ_{μ} are the Pauli matrices.

2. a. Obtain the Heisenberg equation of motion $\frac{\partial\psi}{\partial t}=\frac{i}{\hbar}[H,\psi]$ for the field operator $\psi(x,t)$ in a bosonic system with potential $V(x)$ and Coulomb interaction between the particles. Repeat the calculation for a fermionic system. Is there a difference between the two results?

b. Find the time evolution of the annihilation operators a_i and the creation operators $a_i^{\dagger}$ in a system described by the Hamiltonian $H=\sum_i \varepsilon_i a_i^{\dagger}a_i$. Do it for the bosonic and fermionic cases. Show that the commutation (and anti-commutation) algebra is preserved under the time evolution.

3. The Hubbard model is one of the simplest models of interacting electrons. It describes electrons on a lattice, where each site can hold at most two electrons with opposite spins. The electrons can hop between neighboring sites with an amplitude $-t$ and double occupancy of a single site costs energy U due to the repulsive interaction between the electrons. The model can be solved exactly in one dimension but the two-dimensional model (believed by many to be the minimal model for the high-temperature superconductors) is still an open problem despite decades of effort. Your task is to solve the model on two sites. Let us denote by $a_{i\sigma}$ the annihilation operator of an electron with spin $\sigma=\uparrow,\downarrow$ on site $i=1,2$. The Hamiltonian is: $H=-t\sum_{\sigma}\left(a_{1\sigma}^{\dagger}a_{2\sigma}+a_{2\sigma}^{\dagger}a_{1\sigma}\right)+U\sum_i n_{i\uparrow}n_{i\downarrow}$,

where $n_{i\sigma}=a_{i\sigma}^{\dagger}a_{i\sigma}$ is the particle number operator with spin σ on site i .

a. How many states does the Fock space contain?

b. Find the eigenstates and eigenenergies of the Hamiltonian of system with 0,1,2,3 and 4 electrons.

c. The total spin operator $S_{tot,\mu}=\sum_i S_{i,\mu}$ is a sum of the site spin operators $S_{i,\mu}$, each given by the expression you were asked to derive in 1b. What are the values that $S_{tot,z}$ and $\vec{S}_{tot}^{2}$ take in the ground state of the 2-electron system?

d. Explain the energy and character of the 2-electron ground state in the limit $U \gg t$.

4. Calculate the correlator $\left\langle S_i^z S_j^z \right\rangle$ in the ground state of the spin-$1/2$ XY chain.

5. Consider a ferromagnetic $(J>0)$ Heisenberg chain of N spins of size S: $H=-J\sum_{n=1}^{N}\vec{S}_n\cdot\vec{S}_{n+1}$, where we assume periodic boundary conditions. The spin operators obey the angular momentum commutation algebra: $\left[S_m^\alpha, S_n^\beta\right]=i\,\delta_{m,n}\varepsilon^{\alpha\beta\gamma}S_m^\gamma$.

a. Show that the spin operators can be represented in terms of bosonic operators with the aid of the Holstein-Primakoff transformation:

$$S_n^- = a_n^\dagger\left(2S-a_n^\dagger a_n\right)^{1/2}, \quad S_n^+ = \left(2S-a_n^\dagger a_n\right)^{1/2} a_n, \quad S_n^z = S-a_n^\dagger a_n,$$

where $S^\pm = S^x \pm iS^y$, and $\left[a_m, a_n^\dagger\right]=\delta_{m,n}$, $\left[a_m, a_n\right]=0$.

b. Assume that $S>>1$, and that $\left\langle a_n^\dagger a_n \right\rangle = O(1)$. Expand the transformation for $S^\pm$ to lowest order in $1/S$ and write the Hamiltonian in terms of the bosonic operators within this approximation. Diagonalize it in order to obtain the dispersion of the elementary excitations – the magnons.