Geometrical optics of first-passage functionals of random acceleration

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Random acceleration is a fundamental stochastic process encountered in many applications. In the one-dimensional version of the process a particle is randomly accelerated according to the Langevin equation $\ddot{x}(t) = \sqrt{2D}\xi(t)$, where $x(t)$ is the particle's coordinate, $\xi(t)$ is Gaussian white noise with zero mean, and D is the particle velocity diffusion constant. Here, we evaluate the $A \to 0$ tail of the distribution $P_n(A|L)$ of the functional $I[x(t)] = \int_0^T x^n(t)dt = A$, where T is the first-passage time of the particle from a specified point $x = L$ to the origin, and $n \geqslant 0$. We employ the optimal fluctuation method akin to geometrical optics. Its crucial element is determination of the optimal path—the most probable realization of the random acceleration process $x(t)$, conditioned on specified A, n, and L. The optimal path dominates the $A \to 0$ tail of $P_n(A|L)$. We show that this tail has a universal essential singularity, $P_n(A \to 0|L) \sim \exp(-\frac{\alpha_n L^{3n+2}}{DA^3})$, where α_n is an n-dependent number which we calculate analytically for $n = 0$, 1, and 2 and numerically for other n. For $n = 0$ our result agrees with the asymptotic of the previously found first-passage time distribution.

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I. INTRODUCTION

The random acceleration process is governed by the Langevin equation

$$\ddot{x}(t) = \sqrt{2D}\xi(t). \tag{1}$$

This equation describes the position of a particle moving along the x axis and subject to a random force which is modeled as a Gaussian white noise with zero mean, $\langle \xi(t)\xi(t')\rangle = \delta(t - t')$. Alternatively, $x(t)$ can be considered as the integral of a Brownian motion over time. The random acceleration is a fundamental stochastic process in its own right. On the one hand, it serves as a simple example of a non-Markovian process (which becomes Markovian when considered in two dimensions x and $\dot{x}$; see, e.g., Ref. [1]). On the other hand, its mathematical equivalents have found a variety of applications in physics: from a simplified description of free semiflexible polymer chains in narrow channels [2–7] to interface growth in 1+1 dimensions [8–10] and to decaying turbulence in the Burgers equation [11,12]. In all these systems it is a spatial coordinate which plays the role of time t in Eq. (1), while the polymer shape, or the interface shape, etc., plays the role of x.

Here, we are interested in the statistics of first-passage functionals of the form $I[x(t)] = \int_0^T x^n(t)dt$, defined up to the first-passage time T of the process, starting say at $x = L > 0$, to a specified point in space, for example to the origin. The case $n = 0$ corresponds to the statistics of the first-passage time itself. The case $n = 1$ corresponds to the area under the graph of $x(t)$ until the first passage to the origin. In the context of interface growth, governed by the noisy Mullins-Herring equation [8,9], it describes the area under the stochastic interface until it crosses a zero level in space for the first time. The case $n = 2$ corresponds to the statistics of the moment of inertia of a semiflexible polymer chain of a given length in narrow channels. It is natural then to attempt to calculate the distribution of the values of the first-passage functional $I[x(t)] = \int_0^T x^n(t)dt$ for arbitrary n.

For comparison, the statistics of first-passage Brownian functionals [13,14]—where $x(t)$ is a Brownian motion—is well studied (see Ref. [15] and references therein). For the random acceleration process, however, the problem has been solved only for $n = 0$, that is only for the statistics of the first-passage time itself [6,16,17]. In the absence of general results for the complete distribution $P_n(A|L)$ of the values $I[x(t)] = \int_0^T x^n(t)dt = A$, here we focus on the $A \to 0$ tail of this distribution. We show that this tail exhibits an essential singularity [see Eq. (18) below]. To achieve this goal, we employ the optimal fluctuation method akin to geometrical optics [18]. The method relies on the determination of the optimal path, that is the most likely realization of the process $x(t)$, conditioned on the specified value of $A \to 0$ at given n and L. It is this optimal path that dominates the $A \to 0$ tail of $P_n(A|L)$. Previously, the geometrical optics was applied to a plethora of problems related to the statistics of Brownian motion [15,19–27]. An extension of the method to the random acceleration is a natural next step.

Here is a plan of the remainder of the paper. We complete the formulation of the problem, establish the scaling properties of $P_n(A \to 0|L)$, and derive the governing equation of the optimal fluctuation method in Sec. II. Some analytical and numerical solutions for different n are presented in Sec. III. Section IV includes a brief summary and an extension of our results. A technical derivation is delegated to the Appendix.

II. FORMULATION OF THE PROBLEM AND GOVERNING EQUATIONS

We start by completing the formulation of the problem. The initial and final positions of the particle are

$$x(t = 0) = L, \quad x(T) = 0, \tag{2}$$

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where T is the first-passage time to the origin, and L can be assumed positive without loss of generality. We assume for simplicity that the particle starts with zero velocity:

$$\dot{x}(t=0)=0. \tag{3}$$

We consider first-passage functionals of the form $I[x(t)]=\int_0^T x^n(t)dt$ and study the probability distribution $P_n(A|L)$ of their values A:

$$\int_0^T x^n(t)dt = A. \tag{4}$$

Equations (1)–(4) define the stochastic problem completely. Their dimensional analysis (notice that the units of A depend on n) yields [28] the following *exact* scaling behavior of $P_n(A|L)$,

$$P_n(A|L)=\frac{D^{1/3}}{L^{n+\frac{2}{3}}}F_n\left(\frac{D^{1/3}A}{L^{n+\frac{2}{3}}}\right), \tag{5}$$

with a dimensionless scaling function $F_n(z)$ of the dimensionless argument $z=D^{1/3}AL^{-n-\frac{2}{3}}$. The scaling function $F_n(z)$ is presently unknown. On the physical grounds this function is expected to have a single maximum, of order 1, at $z\sim 1$ (assuming that n is not too close to 0 or not too large). Therefore, the probability distribution $P_n(A|L)$ is expected to have its maximum, of order $D^{1/3}/L^{n+\frac{2}{3}}$, at $A\sim D^{-1/3}L^{n+\frac{2}{3}}$.

Rather than attempting to determine the entire scaling function $F_n(z)$, here we only find its leading-order $z\to 0$ asymptotic. This asymptotic corresponds to the $A\ll D^{-1/3}L^{n+\frac{2}{3}}$ tail of the distribution $P_n(A|L)$ [29]. This large-deviation tail can be obtained by the optimal fluctuation method, akin to geometrical optics. First we need to identify the action functional, corresponding to the Langevin equation (1). We start from the probability distribution of a realization of the white Gaussian noise $\xi(t)$ of unit magnitude (see, e.g., Ref. [14]):

$$\mathcal{P}[\xi(t)]\sim\exp\left[-\frac{1}{2}\int_0^T \xi^2(t)dt\right].$$

Expressing $\xi(t)$ through the particle acceleration $\ddot{x}(t)$ from Eq. (1), we can evaluate the probability distribution of a realization of the random acceleration process in the form of $\sim e^{-S[x(t)]}$ with the action functional

$$S[x(t)]=\frac{1}{4D}\int_0^T \ddot{x}^2(t)dt. \tag{6}$$

The optimal fluctuation method (or geometrical optics) is aimed at finding the "optimal path" $x_*(t)$ that minimizes functional (6) subject to the boundary conditions (2) and (3), to the positivity condition $x(t)>0$ for $0<t<T$, and to the integral constraint

$$I[x(t)]=\int_0^T x^n(t)dt = A. \tag{7}$$

The minimization must be performed not only with respect to different paths $x(t)$, but also with respect to the first-passage time T.

Let us rescale the coordinate, $\tilde{x}=x/L$. The action functional (6) takes the form

$$S[x(t)]=\frac{L^2}{2D}\,s(\tilde{x}),\quad \text{where}\quad s(\tilde{x})=\frac{1}{2}\int_0^T \ddot{\tilde{x}}^2(t)dt. \tag{8}$$

The constraint (7) becomes

$$I[\tilde{x}(t)]=\int_0^T \tilde{x}^n(t)dt=\frac{A}{L^n}. \tag{9}$$

The minimization of the rescaled functional $s(\tilde{x})$ subject to the constraint (9) can be achieved by minimizing the modified functional

$$s_\lambda[\tilde{x}(t)]=s[\tilde{x}(t)]-\lambda I[\tilde{x}(t)]. \tag{10}$$

The Lagrange multiplier λ turns out to be negative, so we can set $\lambda=-\Lambda^4$, where $\Lambda>0$. Now we also rescale time, $\tilde{t}=\Lambda t$. The first-passage time T also gets rescaled, $\tilde{T}=\Lambda T$. The functional (10) becomes

$$s_\lambda[\tilde{x}(\tilde{t})]=\Lambda^3\int_0^{\tilde{T}}\left[\frac{\ddot{\tilde{x}}^2(\tilde{t})}{2}+\tilde{x}^n(\tilde{t})\right]d\tilde{t}, \tag{11}$$

and we will drop the tildes everywhere in the following. Since the rescaled functional $s_0[x(t)]$ (recall that the tildes are dropped) involves the particle acceleration $\ddot{x}(t)$, the Euler-Lagrange equation is of the fourth order (see the Appendix),

$$x^{(4)}(t)+nx^{n-1}(t)=0, \tag{12}$$

where the superscript (4) denotes the fourth derivative with respect to time. Three boundary conditions for Eq. (12) come with the formulation of the original stochastic problem [see Eqs. (2) and (3)]:

$$x(0)=1,\quad \dot{x}(0)=0,\quad \text{and}\quad x(T)=0. \tag{13}$$

The fourth boundary condition,

$$\ddot{x}(T)=0, \tag{14}$$

follows from minimization of the action with respect to all possible variations of the particle velocity $\dot{x}$ at $t=T$ (see the Appendix).

The general solution of the rescaled Euler-Lagrange equation (12) has four arbitrary constants. When this equation is supplemented by the four boundary conditions (13) and (14) [and the inequality $x(0<t<T)>0$], the problem of finding the $A\to 0$ asymptotic of $P_n(A|L)$ is determined completely only for $n=0$ where $A=T$, and one is looking for the distribution $P_n(T|L)$ of first-passage times. For all other $n>0$ one should, in addition, minimize the action $S(A,T)$ with respect to T. The minimization yields the *optimal value* of the first-passage time $T=T_*(A)$ which dominates the probability $P_n(A|L)$ that we are after. As we show in the Appendix, this additional minimization brings about a fifth boundary condition

$$\dddot{x}(T)=0. \tag{15}$$

Once the optimal path $x(t)$ and, for $n\neq 0$, the optimal value $T=T_*(A)$, are found, we can determine Λ from the

relation

$$\Lambda = \frac{L^n}{A}\int_0^{T_*} x^n(t)dt, \tag{16}$$

which follows from the constraint (7) or, equivalently, (9). The original action (6) can now be written as follows,

$$S[x(t)] = \frac{L^2\Lambda^3}{2D}s_0[x(t)], \quad \text{where}$$
$$s_0[x(t)] = \frac{1}{2}\int_0^T \ddot{x}^2(t)dt. \tag{17}$$

Plugging Eq. (16) into the first line of Eq. (17), we obtain, up to a preexponential factor, the $A \to 0$ tail of $P_n(A|L)$. It scales as

$$-\ln P_n(A\to 0|L) \simeq S = \frac{\alpha_n L^{3n+2}}{DA^3}, \tag{18}$$

where

$$\alpha_n = \frac{1}{4}\left[\int_0^{T_*} x^n(t)dt\right]^3 \int_0^{T_*} \ddot{x}^2(t)dt. \tag{19}$$

Equation (18) describes a universal essential singularity $\sim\exp(-A^{-3})$ of the $A \to 0$ tail of the distribution. It is much steeper than the essential singularity $\sim\exp(-A^{-1})$ of the first-passage Brownian functionals [15].

In fact, the large-deviation scaling (18) (with a yet unknown α_n) immediately follows from the exact scaling (5) once we realize that the $A \to 0$ asymptotic of the function $F_n(\cdots)$ in Eq. (5) must exhibit, up to a preexponent, the characteristic weak-noise scaling $F_n \sim \exp(-\Phi/D)$, where Φ depends on A and L but is independent of D. Now let us proceed to find the optimal path, that is to solving Eq. (12) subject to the boundary conditions (13)–(15).

III. SOLUTION

A. General

Equation (12) is easily solvable for $n = 0$, 1, and 2, when the equation is linear. We will present these solutions shortly. In the general case, there is conservation law

$$\dot{x}(t)\dddot{x}(t) - \tfrac{1}{2}\ddot{x}^2(t) + x^n(t) = C = \text{const}, \tag{20}$$

which is a higher-order analog of energy conservation in classical mechanics. The conservation law (20) reduces the order of Eq. (12) by one. Using the boundary conditions (13)–(15) at $t = T$, we find that $C = 0$ for all $n > 0$ [30].

Evaluating the left-hand side of the conservation law (20) (where $C = 0$) at $t = 0$, we uncover one more universal property of the optimal path:

$$\ddot{x}(t=0) = -\sqrt{2} \quad \text{for all} \quad n > 0. \tag{21}$$

Finally, using the conservation law (20) with $C = 0$, integration by parts and Eqs. (13) and (14), we can rewrite the expression (19) for α_n in two equivalent alternative forms:

$$\alpha_n = \frac{1}{6}\left[\int_0^{T_*} x^n(t)dt\right]^3 = \frac{27}{32}\left[\int_0^{T_*} \ddot{x}^2(t)dt\right]^4. \tag{22}$$

B. $n = 0$: First-passage time

The first-passage time distribution $P(T|L)$ of the random acceleration process was determined quite some time ago [6,16,17]. Its short-time asymptotic coincides, in the leading order, with the short-time asymptotic of the propagator of the random acceleration. For the zero initial particle velocity, the exact propagator (see, e.g., Ref. [6]) simplifies to

$$\rho(T, v) = \frac{\sqrt{3}}{2\pi DT^2}\, e^{-\frac{3L^2+3LTv+T^2v^2}{DT^3}}, \tag{23}$$

where $v = \dot{x}(t = T)$ is the particle velocity (in the original units) at $t = T$. We identify the action, corresponding to this distribution,

$$S_\rho(T, v) = \frac{3L^2 + 3LTv + T^2v^2}{DT^3}, \tag{24}$$

and focus on the large-deviation regime $T \to 0$, where this action is much larger than unity. Minimizing $S_\rho(T, v)$ with respect to v, we obtain the optimal value $v_* = -3L/(2T)$. The corresponding minimum of the action,

$$S_\rho(T, v_*) = \frac{3L^2}{4DT^3}, \tag{25}$$

determines the small-A asymptotic of $P(T|L)$,

$$-\ln P(T|L) \simeq \frac{3L^2}{4DT^3}, \tag{26}$$

which obeys our asymptotic scaling relation (18) with $\alpha_0 = 3/4$. Now we will rederive the asymptotic (26) by using the optimal fluctuation formalism.

For $n = 0$ the Euler-Lagrange equation (12) becomes trivial: $x^{(4)} = 0$. Its solution, satisfying the boundary conditions (13) and (14),

$$x(t) = 1 - \frac{3t^2}{2T^2} + \frac{t^3}{2T^3}, \tag{27}$$

is a cubic parabola. Equation (16) yields $\Lambda = 1$. Then, using Eq. (17), we arrive at Eqs. (25) and (26) as to be expected.

C. $n = 1$: First-passage area

For $n = 1$ the Euler-Lagrange equation (12) is still very simple: $x^{(4)} = -1$. Its solution is a quartic parabola. Here, we have to demand all five boundary conditions (13)–(15) which determine the four arbitrary constants and the optimal value of the first-passage time $T_* = 2^{3/4}$. The resulting rescaled optimal path,

$$x(t) = 1 - \frac{t^2}{\sqrt{2}} + \frac{t^3}{3\sqrt[4]{2}} - \frac{t^4}{24}, \tag{28}$$

is depicted, alongside with the optimal acceleration $\ddot{x}(t)$, in Fig. 1. The optimal acceleration is nothing but the (rescaled) optimal realization of the white Gaussian noise $\xi(t)$ [see Eq. (1)]. Needless to say, the optimal realization of the noise looks very differently from a *typical* realization of the noise. Now using Eqs. (18) and (19) for $n = 1$, we obtain

$$-\ln P(A|L) \simeq \frac{108L^5}{625DA^3}, \tag{29}$$

with $\alpha_1 = 108/625$.

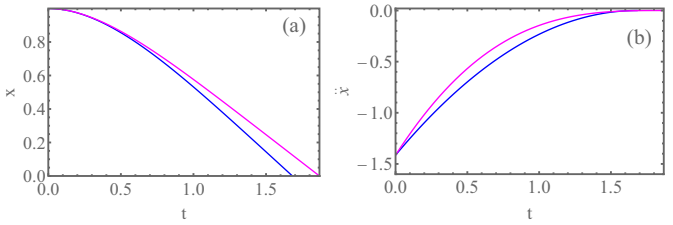

FIG. 1. The rescaled optimal path $x(t)$ (a) and optimal acceleration $\ddot{x}(t)$ (b), dominating the $A \to 0$ asymptotics of $P(A|L)$ for $n = 1$ (blue) and $n = 2$ (magenta). The optimal first-passage time is $T_* = 2^{3/4}$ for $n = 1$ and $T_* = \pi/2^{3/4}$ for $n = 2$.

For completeness, we also present Λ, T_* and the optimal path $x(t)$ in dimensional variables,

$$\Lambda = \frac{3 \cdot 2^{3/4} L}{5A}, \quad T_* = \frac{5A}{3L},$$
$$x(t) = L\left(1 - 2\tau^2 + \frac{4\tau^3}{3} - \frac{\tau^4}{3}\right), \tag{30}$$

where $\tau = t/T_*$.

D. $n = 2$

Here, the Euler-Lagrange equation (12) is still linear and elementary:

$$x^{(4)}(t) + 2x(t) = 0. \tag{31}$$

The solution, obeying the boundary conditions (13)–(15), yields the rescaled optimal path,

$$x(t) = \frac{\left(1 - e^{\frac{t}{T_*}}\right)\sin\left(\frac{\pi t}{2T_*}\right) + \left(1 + e^{\frac{t}{T_*}}\right)\cos\left(\frac{\pi t}{2T_*}\right)}{(1 + e^{-\pi})e^{\frac{\pi t}{2T_*}}}, \tag{32}$$

where $T_* = 2^{-3/4}\pi$ is the optimal first-passage time. Figure 1 shows this optimal path along with the optimal acceleration $\ddot{x}(t)$.

Using Eqs. (18) and (19) for $n = 2$, we obtain

$$-\ln P(A|L) \simeq \frac{27\tanh^4\left(\frac{\pi}{2}\right)L^8}{256DA^3}, \tag{33}$$

Here, $\alpha_2 = (27/256)\tanh^4(\pi/2) = 0.074\,625\ldots$.

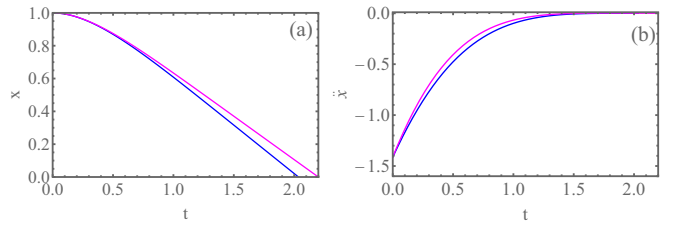

FIG. 2. Numerically found rescaled optimal path $x(t)$ (a) and optimal acceleration $\ddot{x}(t)$ (b), dominating the $A \to 0$ asymptotics of $P(A|L)$ for $n = 3$ (blue) and $n = 4$ (magenta). The optimal first-passage time is $\simeq 2.036$ for $n = 3$ and $T_* \simeq 2.20$ for $n = 4$.

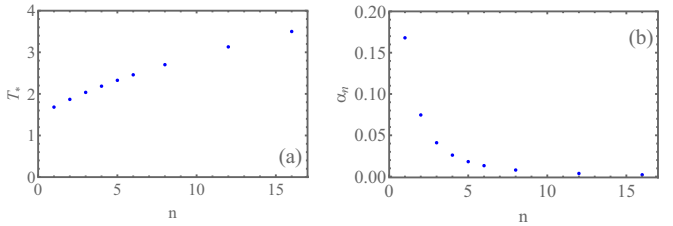

FIG. 3. (a) The optimal first-passage time T_*, conditioned on Eq. (4), (b) and the factor α_n which enters Eqs. (18) and (19), are plotted as functions of n. The points $n = 1$ and $n = 2$ were obtained analytically, and the rest of the points numerically.

In the dimensional variables we have

$$\Lambda = \frac{3L^2\tanh(\pi/2)}{2^{7/4}A}, \quad T_* = \frac{2\pi A\coth(\pi/2)}{3L^2},$$
$$\frac{x(t)}{L} = \frac{(e^{\pi} - e^{\pi\tau})\sin\left(\frac{\pi\tau}{2}\right) + (e^{\pi\tau} + e^{\pi})\cos\left(\frac{\pi\tau}{2}\right)}{1 + e^{\pi}e^{\frac{\pi\tau}{2}}}, \tag{34}$$

where $\tau = t/T_*$.

E. Numerics

For arbitrary n the optimal path can be found numerically. We used artificial relaxation in conjunction with iterations over T. Artificial relaxation was implemented as follows. We introduced artificial time τ and replaced the Euler-Lagrange equation (12) by the fourth-order partial differential equation

$$\partial_\tau X(t,\tau) = -\partial_t^4 X(t,\tau) - nX^{n-1}(t,\tau), \tag{35}$$

where the physical time t plays the role of a coordinate. The sign of the right-hand side of Eq. (35) is chosen so as to enforce relaxation to a steady state, $x(t) = X(t, \tau \to \infty)$, which satisfies our Eq. (12). The initial condition $X(t, \tau = 0)$ is chosen qualitatively similar to the expected steady-state solution. Since we do not know the optimal first-passage time T *a priori*, we use iterations. We first solve Eq. (35) with boundary conditions (13) and (14) for a fixed T (the first guess of T_*) until the steady-state solution $x(t)$ is reached. Then we evaluate the third derivative $\partial_t^{(3)} X(t, \tau \gg 1)$ at $t = T$, and iterate T until the third derivative vanishes [as Eq. (15) demands] with desired accuracy. Alternatively, one can iterate until $\partial_t^2 X(t, \tau \gg 1)$ at $t = 0$ approaches $-\sqrt{2}$ [see Eq. (21)]. We validated the method by comparing the numerically found $x(t)$ with the analytical solutions for $n = 1$ and 2. The accuracy was monitored by checking the conservation law (20) with $C = 0$. Once T_* and $x(t)$ are found, we can evaluate α_n from any of the equations (19) or (22). We used a standard partial differential equation solver of *Mathematica* [31].

Figure 2 shows the numerically found optimal paths $x(t)$ and the optimal accelerations $\ddot{x}(t)$ for $n = 3$ and 4. In these cases $T_* \simeq 2.036$ and 2.185, respectively, whereas the $A \to 0$ asymptotics of $P(A|L)$ are described by Eq. (18) with $\alpha_3 \simeq 0.041$ and $\alpha_4 \simeq 0.026$. Overall, we solved the problem numerically and found the optimal first-passage time T_* and the factor α_n for a range of n (see Fig. 3). As one can see, T_* increases with n, while α_n decreases.

IV. SUMMARY AND DISCUSSION

Statistics of first-passage functionals provide a useful characterization of random processes. Here, we evaluated the $A \to$

0 tail of these statistics for the random acceleration process. We also used this problem to extend the optimal fluctuation (or geometrical optics) method to a stochastic process of a higher order. In addition to the $A \to 0$ asymptotic of the probability distribution $P_n(A|L)$, we calculated analytically and numerically the optimal paths of the conditioned processes at different n. These provide an interesting insight into the nature of large deviations in this system.

The geometrical-optics calculations can be extended to the case where the particle velocity at $t = 0$ is nonzero: $v_0 \neq 0$. The more interesting case here is $v_0 < 0$, when the particle arrives at the origin with probability 1. Here a new effect appears: The particle can reach the origin *deterministically* along the ballistic trajectory $x(t) = L - |v_0|t$. The expected first-passage time (not subject to any additional constraint) is $\bar{T} = L/|v_0|$. Evaluating the functional $I[x(t)]$ in Eq. (7) on the ballistic trajectory, we obtain

$$A \equiv \bar{A} = \frac{L^{n+1}}{(n+1)|v_0|}. \tag{36}$$

If we condition the process on $A < \bar{A}$ or $A > \bar{A}$, the optimal path will be different from the deterministic one, and it can be found with the same formalism we used, except that the second condition in Eq. (13) should be replaced by $\dot{x}(t = 0) = -|v_0|$ (in the dimensional variables). The resulting action (6) vanishes at $A = \bar{A}$. In a small vicinity around $A = \bar{A}$ the action is quadratic in $A - \bar{A}$. For such small fluctuations of A the probability distribution is approximately Gaussian. Applicability of geometrical optics requires that the resulting action be much larger than unity.

The simplest example of such a calculation is the evaluation of the first-passage time distribution itself, that is $n = 0$. Here, instead of Eq. (27) we obtain

$$x(t) = L + v_0 t - \frac{3(L + v_0 T)t^2}{2T^2} + \frac{(L + v_0 T)t^3}{2T^3}. \tag{37}$$

The resulting probability density $P(T, L, v_0 < 0)$ is given, up to a preexponential factor, by the expression

$$-\ln P(T, L, v_0 < 0) \simeq S = \frac{3(L - |v_0|T)^2}{4DT^3}, \tag{38}$$

which agrees with the short-time asymptotic of the exact propagator of the random acceleration for nonzero initial particle velocity [6]. As to be expected, the action (38) vanishes at $T = \bar{T} = L/|v_0|$. For $v_0 = 0$, Eq. (38) coincides with Eq. (25), again as to be expected.

Finally, the problem of statistics of the first-passage functionals $I[x(t)] = \int_0^T x^n(t)dt$ can be extended to a whole family of processes, described by the Langevin equation $d^k x(t)/dt^k = \sqrt{2D}\xi(t)$, where k is any positive integer. The cases of $k = 1$ and $k = 2$ correspond to the Brownian motion and random acceleration, respectively. Again, let $x(0) = L$, and again suppose for simplicity that all the derivatives of $x(t)$ with order less than k vanish at $t = 0$. Then the exact scaling behavior of probability distribution $P_n^{(k)}(A|L)$ of the values $I[x(t)] = A$ follows from dimensional analysis,

$$P_n^{(k)}(A|L) = \frac{D^\nu}{L^{n+2\nu}} F_n^{(k)}\left(\frac{D^\nu A}{L^{n+2\nu}}\right), \tag{39}$$

where $F_n^{(k)}(z)$ is an unknown scaling function, and $\nu = 1/(2k-1)$. In its turn, the leading-order $A \to 0$ asymptotic of $P_k(A|L)$ must have the characteristic weak-noise form

$$-\ln P_n^{(k)}(A \to 0|L) \simeq \frac{\alpha_n^{(k)} L^{\frac{n}{\nu}+2}}{DA^{1/\nu}}, \tag{40}$$

where $\alpha_n^{(k)}$ is a numerical factor which depends on k and n. As one can see from Eq. (40), for all these models theory predicts am essential singularity at $A \to 0$, and the singularity becomes stronger as k is increased.

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APPENDIX: DERIVATION OF EQ. (12) AND BOUNDARY CONDITION (15)

Here, we temporarily switch back to the original variables and consider a linear variation of the constrained action functional

$$s_\lambda[x(t), T] = \int_0^T \left(\frac{\ddot{x}^2}{2} - \lambda x^n\right)dt, \tag{A1}$$

with respect to small variations of both $x(t)$ and T: $x(t) \to x(t) + \delta x(t)$ and $T \to T + \delta T$. We need to linearize the variation

$$\delta s_\lambda = s[x(t) + \delta x(t), T + \delta T] - s[x(t), T] \tag{A2}$$

with respect to δx and δT. The linearization yields, after simple algebra,

$$\delta s_\lambda = \int_0^T (\ddot{x}\delta\ddot{x} - \lambda n x^{n-1}\delta x)dt + \int_T^{T+\delta T}\left(\frac{\ddot{x}^2}{2} - \lambda x^n\right)dt. \tag{A3}$$

Performing two integrations in parts in the first integral, evaluating the second integral in the limit of $\delta T \to 0$, and taking into account the boundary conditions (13), we obtain

$$\delta s_\lambda = \int_0^T (x^{(4)} - \lambda n x^{n-1})\delta x\, dt + \ddot{x}(T)\delta\dot{x}(T) + \left[\frac{\ddot{x}^2(T)}{2} - \lambda x^n(T) - \dddot{x}(T)\dot{x}(T)\right]\delta T. \tag{A4}$$

Each of the three terms in the variation must vanish independently for arbitrary δx and δT. The first term yields the Euler-Lagrange equation $x^{(4)} - \lambda n x^{n-1} = 0$ which, upon the rescaling $\Lambda t \to t$ (we recall that $\lambda \equiv -\Lambda^4$), coincides with Eq. (12). The second term yields the boundary condition (14). Using the latter condition and the condition $x(T) = 0$ in the third term, we arrive at the condition $\ddot{x}(T)\dot{x}(T) = 0$. Now we have to choose one of two options: $\dot{x}(T) = 0$ or $\dddot{x}(T) = 0$ (they cannot hold simultaneously, as the problem then would be overdetermined). As one can check [32], the condition $\dot{x}(T) = 0$ would give a local *maximum* of the action functional $S[x(t)]$ as a function of T, whereas the condition $\ddot{x}(T) = 0$ yields the desired minimum.

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[28] The probability distribution $P_n(A|L)$ depends only on the dimensional parameters A, D, and L and on the dimensionless parameter n. In its turn, the dimensions of parameters A, D, and L include only two parameters with independent dimensions: length and time. By the Buckingham Π theorem (see, e.g., Ref. [34]), there is only one independent dimensionless combination of A, D, and L, and it is natural to choose the combination $D^{1/3}AL^{-n-\frac{2}{3}}$, proportional to A. Then, using the normalization condition $\int_0^\infty P_n(A|L)dA = 1$, one arrives at Eq. (5). See Ref. [34] for many instructive examples of dimensional analysis in physics.

[29] At very short times, the random acceleration model (where a real stochastic force is modeled by white noise) breaks down, and some additional, "microscopic" physics should be taken into account. Therefore, as it often occurs in physics, our small-A results should be considered as an intermediate asymptotic: They are expected to be applicable when A is much smaller than $D^{-1/3}L^{n+2/3}$, but much larger than a (system-specific) microscopic timescale where the random acceleration model breaks down.

[30] By introducing a new function $z(x) = [-\dot{x}(t)]^{3/2}$, Eq. (20) can be reduced to a second-order equation

$$z''(x) = -\tfrac{3}{2}x^n z^{-5/3}(x),$$

which is known as the Emden-Fowler equation [35]. Unfortunately, its exactly soluble cases do not seem to extend beyond $n = 0$, 1, and 2, where it is much easier to solve Eq. (12) directly.

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