Dynamic Response of One-Dimensional Interacting Fermions

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We evaluate the dynamic structure factor $S(q, \omega)$ of interacting one-dimensional spinless fermions with a nonlinear dispersion relation. The combined effect of the nonlinear dispersion and of the interactions leads to new universal features of $S(q, \omega)$. The sharp peak $S(q, \omega) \propto q\delta(\omega - uq)$, characteristic for the Tomonaga-Luttinger model, broadens up; $S(q, \omega)$ for a fixed q becomes finite at arbitrarily large ω. The main spectral weight, however, is confined to a narrow frequency interval of the width $\delta\omega \sim q^2/m$. At the boundaries of this interval the structure factor exhibits power-law singularities with exponents depending on the interaction strength and on the wave number q.

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Low-energy properties of fermionic systems are sensitive to interactions between fermions. The effect of interactions is the strongest in one dimension (1D), where single-particle correlation functions exhibit power-law singularities, in a striking departure from the behavior in higher dimensions. Much of our current understanding of 1D fermions is based on the Tomonaga-Luttinger (TL) model [1]. The crucial ingredient of the model is the assumption of a *strictly linear* fermionic dispersion relation. The TL model, often used in conjunction with a powerful bosonization technique [2], allows one to evaluate various correlation functions, such as momentum-resolved [3,4] and local [5] single-particle densities of states.

Unlike the single-particle correlation functions, the two-particle correlation functions of the TL model exhibit behavior rather compatible with that expected for a Fermi liquid with the linear spectrum of quasiparticles. For example, the dynamic structure factor (the density-density correlation function)

$$S(q, \omega) = \int dxdte^{i(\omega t - qx)}\langle \rho(x, t)\rho(0, 0)\rangle \tag{1}$$

at small q takes the form [3] $S_{\rm TL}(q, \omega) \propto q\delta(\omega - uq)$. It means that the quanta of density waves propagating with plasma velocity u are true eigenstates of the TL model; these bosonic excitations have an infinite lifetime.

Below we show that such a simple behavior is an artifact of the linear spectrum approximation. In reality, the spectrum of fermions always has some nonlinearity,

$$\xi_{R/L,k} = \pm vk + k^2/2m + \cdots, \tag{2}$$

where the upper (lower) sign corresponds to the right (R) [left (L)] movers, and $k = p \mp p_F$ are momenta measured from the Fermi points $\pm p_F$. [For Galilean-invariant systems the expansion (2) terminates at k^2.]

The finite curvature ($1/m \neq 0$) affects drastically the functional form of $S(q, \omega)$. In a clear deviation from the results of TL model, power-law singularities now arise not only in the single-particle correlation functions, but in the structure factor as well. We show that the singularities in these two very different objects have a common origin, proliferation of low-energy particle-hole pairs, and evaluate the corresponding exponents.

Because of the success of the bosonization technique [2], it is tempting to treat the spectrum nonlinearity as a weak interaction between the TL bosons. Indeed, the nonlinearity gives rise to a three-boson interaction with the coupling constant $\propto 1/m$ [6]. However, attempts to treat this interaction perturbatively fail as the finite-order contributions to the boson's self-energy diverge at the mass shell [7]. A reliable method of resumming the corresponding series is yet to be developed.

We found it more productive to approach the problem from the fermionic perspective. One has then a benchmark reference point: the structure factor of free fermions. At zero temperature, the structure factor has a simple physical meaning of the absorption rate of a photon with energy ω and momentum q [8]. Without interaction, absorption of a photon results in a creation of a single particle-hole pair. For a fixed $q < 2p_F$, the energy of this pair lies within the interval

$$\omega_- < \omega < \omega_+, \qquad \omega_\pm = uq \pm q^2/2m \tag{3}$$

(for free fermions $u = v$). The bounds ω_+ (ω_-) correspond to particle-hole pairs in which a hole (a particle) is created just below (just above) the Fermi energy. If ω is outside the interval (3), the energy and momentum conservation laws cannot be satisfied and the structure factor vanishes. Within this interval, $S(q, \omega)$ is independent of ω,

$$S(q, \omega) = S_0(q) = m/q, \qquad \omega_- < \omega < \omega_+; \tag{4}$$

see Fig. 1. Accordingly, $S(q, \omega)$ at a fixed q exhibits a peak which has a "rectangular" shape with the width $\delta\omega = \omega_+ - \omega_- = q^2/m$ [10].

If even a weak interaction is now turned on, its effect on the structure factor is dramatic; see Fig. 1. The structure factor still vanishes below the (renormalized) lower ab-

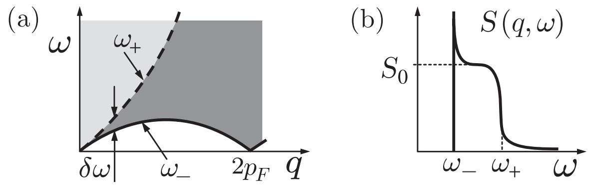

FIG. 1. (a) Without interactions, the structure factor differs from zero only for $\omega_- < \omega < \omega_+$. In the presence of interactions, $S \neq 0$ at $\omega > \omega_+$ as well. For repulsive interactions, $S(q, \omega)$ has a power-law divergence at $\omega = \omega_-(q)$ with an exponent depending on q. (b) Sketch of $S(q, \omega)$ at a fixed small $q \ll p_F$; see Eqs. (5) and (7).

sorption "edge" $\omega = \omega_-$, but above the edge $S(q, \omega)$ develops a power-law singularity,

$$\frac{S(q, \omega)}{S_0(q)} = \left(\frac{\delta\omega}{\omega - \omega_-}\right)^{\mu}, \qquad 0 < \omega - \omega_- \ll \delta\omega. \quad (5)$$

The exponent μ here is a smooth function of q,

$$\mu(q) = \frac{m}{\pi q}(V_0 - V_q) \quad (6)$$

[V_p is the Fourier component of the intrabranch interaction potential; see Eq. (10) below]. The divergence in Eq. (5) has the same origin as the familiar x-ray edge singularity in metals [11].

In the presence of interactions, absorption of a photon is accompanied by the creation of multiple particle-hole pairs, which allows the conservation laws to be satisfied at arbitrarily high ω [9], so that $S(q, \omega) \neq 0$ for $\omega > \omega_+$. However, at $\omega = \omega_+$ the structure factor still exhibits a power-law singularity. At $|\omega - \omega_+| \ll \delta\omega$ we find

$$\frac{S(q, \omega)}{S_0(q)} = \begin{cases} [\frac{\omega_+ - \omega}{\delta\omega}]^{\mu} + \frac{\nu}{\mu}, & \omega < \omega_+ \\ \frac{\nu}{\mu}(1 - [\frac{\omega - \omega_+}{\delta\omega}]^{\mu}), & \omega > \omega_+ \end{cases} \quad (7)$$

with

$$\nu(q) = \left(\frac{q}{4mu}\right)^2 \left(\frac{U_0}{2\pi u}\right)^2 \ll \mu(q). \quad (8)$$

Here U_0 is the interaction between the right and left movers; see Eq. (10) below.

Finally, at high frequencies $S(q, \omega)$ can be evaluated [12] in the second order of perturbation theory in the interaction between fermions,

$$S(q, \omega) = 2\nu \frac{uq^2}{\omega^2 - u^2 q^2}, \qquad \omega - \omega_+ \gg \delta\omega. \quad (9)$$

Note that $S \propto q^4$ in Eq. (9), as expected for a multipair contribution to the photon absorption rate [9]. Therefore, for $q \ll p_F$ the high-frequency "tail" yields a negligible contribution to the f-sum rule: the main spectral weight of $S(q, \omega)$ is still confined within the narrow frequency window (3). At the borders of this interval $S(q, \omega)$ develops power-law singularities; see Eqs. (5) and (7). The resulting rather peculiar shape of the peak in $S(q, \omega)$ is sketched in Fig. 1(b). It is certainly very different from a simple Lorentzian assumed in, e.g., Ref. [13]. If one were to interpret the finite width of the peak as a lifetime of the TL bosons, one would conclude that the boson's decay is manifestly nonexponential. Instead, it is governed by power laws, indicating strong nonlinearity-induced correlations between the TL bosons.

Equations (5)–(8) represent the main result of this Letter. We now outline their derivation. We describe spinless 1D fermions by the Hamiltonian

$$H = \sum_{\alpha,k} \xi_{\alpha,k} \psi^\dagger_{\alpha,k} \psi_{\alpha,k} + \frac{1}{2L} \sum_{p \neq 0} \left\{ V_p \sum_{\alpha} \rho_{\alpha,p} \rho_{\alpha,-p} + 2U_p \rho_{R,p} \rho_{L,-p} \right\}, \quad (10)$$

where $\alpha = R, L$, the single-particle energies $\xi_{\alpha,k}$ are given by Eq. (2), $\rho_{\alpha,p} = \sum_k \psi^\dagger_{\alpha,k-p} \psi_{\alpha,k}$, and L is size of the system. We consider short-range interaction, so that V_0, U_0 are finite and $V_0 - V_p \propto p^2$ (and similarly for U_p) for small $p \ll p_F$. (Note that $V_p = U_p$ for Galilean-invariant systems.)

Let us concentrate on the immediate vicinity of the lower absorption edge, $\omega \to \omega_-$. Divergent corrections to $S(q, \omega)$ appear already in the first order in the interaction strength; see the diagrams (a) and (b) in Fig. 2. Evaluation of these contributions (which differ from zero only at $\omega_- < \omega < \omega_+$) yields

$$\frac{\delta S(q, \omega)}{S_0(q)} = \mu \ln\left[\frac{\delta\omega}{\omega - \omega_-}\right], \qquad 0 < \omega - \omega_- \ll \delta\omega \quad (11)$$

with μ given by Eq. (6). It is not difficult to pinpoint the origin of the logarithmic divergencies in the perturbation theory. Absorption of a photon results in the creation of a "deep" hole at $k \approx -q$. Particles near the Fermi level may then scatter off the hole with a small ($\ll q$) momentum transfer. Excitation of multiple low-energy particle-hole pairs then leads to the power-law enhancement of the absorption rate, similar to the edge singularity in the x-ray absorption spectra in metals [11]. Unlike the conventional x-ray singularity problem, in our case the deep hole

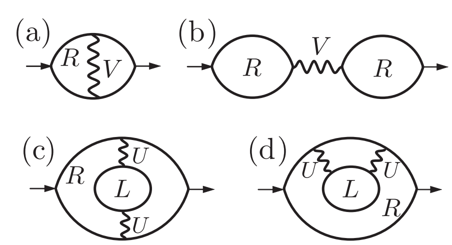

FIG. 2. (a), (b) First-order correction to $S(q, \omega)$, logarithmically divergent at $\omega \to \omega_- + 0$ and $\omega \to \omega_+ - 0$. Each arc denotes a Green function for right-movers. (c), (d) Second-order contributions that diverge logarithmically at $\omega \to \omega_+ + 0$; internal loops correspond to left-movers.

is mobile. It is known, however, that in 1D the edge singularity remains intact even when the dynamics of the hole is taken into account [14].

The above perturbation theory analysis and the analogy with the x-ray singularity suggest that the most divergent terms of the perturbative expansion can be summed by replacing the original model Eq. (10) with an appropriate effective Hamiltonian [11]. It should include *two* narrow (of the width $k_0 \ll q$) strips of states: one around the (right) Fermi point $k = 0$, and another around $k = -q$. The former accommodates low-energy particle-hole pairs, while the later hosts a deep hole. The corresponding effective Hamiltonian H_- is then obtained by projecting Eq. (10) onto the states of right-movers with $|k| < k_0 \ll q$ and $|k + q| < k_0$ (the r and d subbands in Fig. 3), while states outside these intervals are regarded as either empty or occupied.

Furthermore, for $k_0 \ll q$ the spectrum within the two subbands can be linearized. Using

$$\psi_r(x) = \sum_{|k|<k_0} \frac{e^{ikx}}{\sqrt{L}} \psi_{R,k}, \qquad \psi_d(x) = \sum_{|k+q|<k_0} \frac{e^{i(k+q)x}}{\sqrt{L}} \psi_{R,k},$$

we write the projected Hamiltonian in the coordinate representation [15],

$$H_- = \int dx \psi_r^\dagger (-iu_0 \partial_x) \psi_r + \int dx \psi_d^\dagger (-\omega_- - iu_{-q} \partial_x) \psi_d - (V_0 - V_q) \int dx \rho_d(x) \rho_r(x). \tag{12}$$

Here $\rho_r(x) =: \psi_r^\dagger(x)\psi_r(x)$: and $\rho_d(x) = \psi_d(x)\psi_d^\dagger(x)$ are densities of particles and holes in the corresponding subbands (the colons denote the normal ordering). The velocities u_0 and u_{-q} are given by

$$u_p = v + V_0/2\pi + p/m \tag{13}$$

with $p = 0, -q$, and include corrections due to both the interaction and the spectrum nonlinearity (we neglected $V_p - V_0 \propto p^2$ here). Finally, ω_- is the lower absorption edge given by Eq. (3) with $u = u_0$.

In terms of the effective Hamiltonian (12), the structure factor Eq. (1) takes the form

$$S(q, \omega) = \int dx dt e^{i\omega t} \langle B(x, t) B^\dagger(0, 0) \rangle. \tag{14}$$

The operator $B^\dagger = \psi_r^\dagger \psi_d$ creates an intersubband particle-hole pair (an exciton). With the effective Hamiltonian (12), the correlation function (14) can be evaluated by known methods [14,16]. Indeed, the total number of d holes $N_d = \int dx \rho_d(x)$ commutes with H_-. Since the entire d subband lies below the Fermi level, the ground state of H_- corresponds to $N_d = 0$. The operator $B^\dagger$ in Eq. (14) creates one d hole, which propagates until it is annihilated by the operator B. Therefore, as far as the evaluation of Eq. (14) is concerned, H_- can be simplified even further by replacing $\psi_d(x) \to \mathcal{P}\psi_d(x)\mathcal{P}$, where $\mathcal{P}$ is a projector onto states with $N_d = 0, 1$. It is easy to see that the projected operators satisfy

$$\psi_d(y)\rho_d(x) = 0, \qquad \rho_d(x)\psi_d(y) = \delta(x - y)\psi_d(y),$$

from which it follows that $[\rho_d(x), \rho_d(y)] = 0$.

We now bosonize ψ_r field in (12) according to [2]

$$\psi_r(x) = \sqrt{k_0} e^{i\varphi(x)}, \qquad [\varphi(x), \varphi(y)] = i\pi \mathrm{sgn}\,(x - y), \tag{15}$$

which yields $H_- = H_0 + \delta H$, where

$$H_0 = \frac{u_0}{4\pi} \int dx (\partial_x \varphi)^2 + \int dx \psi_d^\dagger (-\omega_- - iu_{-q} \partial_x) \psi_d,$$
$$\delta H = -\frac{1}{2\pi}(V_0 - V_q) \int dx \rho_d(x) \partial_x \varphi. \tag{16}$$

The Hamiltonian H_- can be diagonalized by the unitary transformation [16] $\tilde{H}_- = e^{iW} H_- e^{-iW}$ with

$$W = \theta \int dx \rho_d(x) \partial_x \varphi, \qquad \theta = \frac{1}{2\pi} \frac{V_0 - V_q}{u_{-q} - u_0} = -\frac{\mu}{2},$$

where μ is given by Eq. (6). To the linear order in V, the transformation yields $\tilde{H}_- = H_0$. At the same time, the transformation modifies the operator $B^\dagger$,

$$\tilde{B}^\dagger(x) = e^{iW} B^\dagger(x) e^{-iW} = \sqrt{k_0} e^{-i(1-\mu/2)\varphi} \psi_d. \tag{17}$$

Since $\tilde{H}_- = H_0$ is quadratic, evaluation of Eq. (14) is straightforward. The structure factor vanishes identically at $\omega < \omega_-$, while at $\omega > \omega_-$ it is given by Eq. (5), valid with logarithmic accuracy [i.e., up to a numerical factor in the large parentheses in Eq. (5)].

A similar procedure can be employed to calculate the structure factor near the upper edge $\omega = \omega_+$. At $\omega \to \omega_+ - 0$ the first order in V correction to $S(q, \omega)$ [see Figs. 1(a) and 1(b)] diverges,

$$\frac{\delta S(q, \omega)}{S_0(q)} = \mu \ln\left[\frac{\omega_+ - \omega}{\delta\omega}\right], \qquad 0 < \omega_+ - \omega \ll \delta\omega. \tag{18}$$

The nonvanishing contributions to $S(q, \omega)$ at $\omega > \omega_+$ appear in the second order in U; see diagrams (c) and (d) in Fig. 2. These diagrams describe a process in which an absorption of a photon results in the final state that has two particle-hole pairs on the opposite branches of the Fermi surface [12]. The corresponding contribution reads

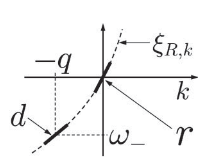

FIG. 3. The states of right-movers included in the effective Hamiltonian (12).

$$\frac{\delta S(q,\omega)}{S_0(q)} = \nu \ln\left(1+\frac{\delta\omega}{\omega-\omega_+}\right)\frac{2\omega_+}{\omega+\omega_+}, \qquad \omega > \omega_+ \tag{19}$$

with ν given by Eq. (8). Equation (19) reduces to Eq. (9) at $\omega - \omega_+ \gg \delta\omega$, and diverges logarithmically at $\omega \to \omega_+$.

As above, the divergent contributions can be summed up by replacing the original model Eq. (10) with an appropriate effective Hamiltonian. In this case the d subband lies well above the Fermi level (near $k = q$), and contains at most a single high-energy particle. However, unlike at $\omega \to \omega_-$, the interaction with the left-movers now has to be explicitly taken into account [17]. The counterpart of Eq. (12) then reads

$$H_+ = \int dx\{iu_0(\psi_l^\dagger \partial_x \psi_l - \psi_r^\dagger \partial_x \psi_r) + \psi_d^\dagger(\omega_+ - iu_q\partial_x)\psi_d\} + \int dx\{(V_0 - V_q)\rho_d\rho_r + U_0(\rho_r + \rho_d)\rho_l\}, \tag{20}$$

where the field operators are defined similarly to $\psi_{r,d}$ in Eq. (12), and ρ_d, ρ_r, and ρ_l are the normal-ordered particle densities. The structure factor is given by Eq. (14) with the exciton creation operator $B^\dagger = \psi_d^\dagger \psi_r$. Similar to the above, after bosonizing the r and l subbands, the effective Hamiltonian H_+ can be diagonalized by means of a unitary transformation. A straightforward, although lengthy calculation [18] then yields Eq. (7) for the structure factor. Note that at $\omega \to \omega_+$ the edge singularity leads to the *suppression* of the absorption rate. Technically, the difference with Eq. (5) arises because, unlike d holes in Eq. (12), d particles in Eq. (20) move *faster* than the particles near the Fermi level: $u_q - u_0 = q/m > 0$.

The above consideration, leading to power-law singularities in $S(q,\omega)$, is limited to zero temperature $T = 0$. However, as long as temperature remains low, $T \ll \delta\omega$, its main effect is to cut off the power-law singularities. This amounts to the replacement $|\omega - \omega_\pm| \to \max\{\pi T, |\omega - \omega_\pm|\}$ in Eqs. (5) and (7).

In conclusion, we found the dynamic structure factor $S(q,\omega)$ of interacting fermions in one dimension, without resorting to the TL model. This allowed us to uncover certain universal features in the behavior of $S(q,\omega)$. The structure factor has a threshold, $\omega = \omega_-(q)$, stemming from kinematic constraints; see Fig. 1. The divergence of S at the threshold, Eq. (5), is characterized by exponent μ, which is a smooth function of the wave number q. For weak interactions, the explicit form of the function $\mu(q)$, valid at any $q < 2p_F$, is given in Eq. (6). We believe, however, that the appearance of the power-law divergency in S along the entire boundary $\omega = \omega_-(q)$ is a generic feature, not limited to weak interactions. Indeed, the methods we employed to arrive at Eq. (5) rely on smallness of $\mu(q)$, which is achieved at sufficiently small q at any strength of interactions V and U. Also, at $q \to 2p_F$ the power-law divergence of S is evident from the conventional Luttinger liquid theory [2]. Remarkably, the origin of the threshold singularity in S can be traced back to the physics of Mahan exciton in theory of x-ray edge singularity [11], and remains universal at any q. The same physics dictates the existence of a power-law singularity in $S(q,\omega)$ at $\omega = \omega_+(q)$; see Eq. (7).

Besides being of a fundamental interest, the knowledge of the structure factor with nonlinear dispersion relation is crucial for understanding the Coulomb drag, photovoltaic effect, and other phenomena that owe their existence to the particle-hole asymmetry. The developed theory is also applicable to the inelastic neutron scattering off antiferromagnetic spin chains [2,19].

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[1] S. Tomonaga, Prog. Theor. Phys. **5**, 544 (1950); J. M. Luttinger, J. Math. Phys. (N.Y.) **4**, 1154 (1963); D. C. Mattis and E. H. Lieb, J. Math. Phys. (N.Y.) **6**, 304 (1965).

[2] T. Giamarchi, *Quantum Physics in One Dimension* (Oxford University Press, New York, 2004); A. O. Gogolin, A. A. Nersesyan, and A. M. Tsvelik, *Bosonization and Strongly Correlated Systems* (Cambridge University Press, Cambridge, England, 1998).

[3] I. E. Dzyaloshinskii and A. I. Larkin, Zh. Eksp. Teor. Fiz. **65**, 411 (1973) [Sov. Phys. JETP **38**, 202 (1974)].

[4] A. Luther and I. Peschel, Phys. Rev. B **9**, 2911 (1974).

[5] C. L. Kane and M. P. A. Fisher, Phys. Rev. Lett. **68**, 1220 (1992); Phys. Rev. B **46**, 15 233 (1992).

[6] F. D. M. Haldane, J. Phys. C **14**, 2585 (1981).

[7] K. V. Samokhin, J. Phys. Condens. Matter **10**, L533 (1998).

[8] The structure factor Eq. (1) is related to the imaginary part of the corresponding retarded correlation function by the Fluctuation-Dissipation Theorem; see [9].

[9] P. Nozières and D. Pines, *The Theory of Quantum Liquids* (Perseus Books, Cambridge, MA, 1999).

[10] In the bosonic representation, even the simple result $\delta\omega \sim q^2/m$ is nonperturbative: the first order in $1/m$ correction to S vanishes, while the next one diverges [7].

[11] G. D. Mahan, *Many-Particle Physics* (Springer, New York, 2000).

[12] M. Pustilnik, E. G. Mishchenko, L. I. Glazman, and A. V. Andreev, Phys. Rev. Lett. **91**, 126805 (2003).

[13] P. Pirooznia and P. Kopietz, cond-mat/0512494.

[14] T. Ogawa, A. Furusaki, and N. Nagaosa, Phys. Rev. Lett. **68**, 3638 (1992); L. Balents, Phys. Rev. B **61**, 4429 (2000).

[15] The projection transforms the interaction term in the Hamiltonian Eq. (10) into the form $V_0\psi_d^\dagger\psi_d\psi_r^\dagger\psi_r + V_q\psi_d^\dagger\psi_r\psi_r^\dagger\psi_d$, which after the normal ordering yields the last term in Eq. (12).

[16] K. D. Schotte and U. Schotte, Phys. Rev. **182**, 479 (1969).

[17] Inclusion of the interaction with the left-movers in the effective Hamiltonian Eq. (12) would lead to a second order in U correction to the exponent μ [18].

[18] M. Khodas *et al.* (to be published).

[19] G. Müller, H. Thomas, H. Beck, and J. Bonner, Phys. Rev. B **24**, 1429 (1981).