Notes on Analytical Mechanics

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Here I review some of the most basic notions of analytical mechanics. This note is not to be viewed as a replacement of a regular course but rather a summary of basic results I rely upon in the E&M class. I often refer to the classical trajectory and I mean by this the trajectory that satisfies the equation of motion, and therefore is actually realized.

I. LAGRANGEAN FORMULATION AND EULER-LAGRANGE EQUATIONS

For the set of generalized coordinates, q_i the Lagrangean, $L(q_i, \dot{q}_i, t)$ defines the action $S = \int_{t_1}^{t_2} dt L(q_i, \dot{q}_i, t)$ that is at minimum for the classical trajectory $q_i^c(t)$. The minimization of action subject to the condition,

$$\delta q_i(t_1) = \delta q_i(t_2) = 0 \tag{1}$$

implies

$$0 = \delta S\left[\delta q_i(t) \equiv q_i(t) - q^c(t)\right] = \int_{t_1}^{t_2} \left\{L[q_i^c + \delta q_i(t), \dot{q}_i^c + \delta \dot{q}_i(t)] - L[q_i^c, \dot{q}_i^c]\right\} \tag{2}$$

We perform the expansion in (2)

$$\left\{L[q_i^c + \delta q(t), \dot{q}_i^c + \delta \dot{q}_i(t)] - L[q_i^c, \dot{q}_i^c]\right\} = \sum_i \left.\frac{\partial L}{\partial q_i}\right|_{q^c} \delta q_i + \sum_i \left.\frac{\partial L}{\partial \dot{q}_i}\right|_{q^c} \delta \dot{q}_i \tag{3}$$

Now notice the obvious relationship,

$$\delta \dot{q}_i = \frac{dq_i(t)}{dt} - \frac{dq_i^c(t)}{dt} = \frac{d}{dt}\left[q_i(t) - q_i^c(t)\right] = \frac{d}{dt}\delta q_i(t) \tag{4}$$

Therefore, we have

$$\delta S\left[\delta q_i(t)\right] = \sum_i \int_{t_1}^{t_2} dt \left.\frac{\partial L}{\partial q_i}\right|_{q^c} \delta q_i + \sum_i \int_{t_1}^{t_2} dt \left.\frac{\partial L}{\partial \dot{q}_i}\right|_{q^c} \delta \dot{q}_i \tag{5}$$

Using (4) and integrating by parts in the second term and using (1) we get

$$\sum_i \int_{t_1}^{t_2} dt \left.\frac{\partial L}{\partial \dot{q}_i}\right|_{q^c(t)} \delta \dot{q}_i = \sum_i \int_{t_1}^{t_2} dt \left.\frac{\partial L}{\partial \dot{q}_i}\right|_{q^c(t)} \frac{d}{dt}\delta q_i(t) = \sum_i \underbrace{\int_{t_1}^{t_2} dt \frac{d}{dt}\left\{\left.\frac{\partial L}{\partial \dot{q}_i}\right|_{q^c(t)} \delta q_i(t)\right\}}_{=0, Eq.~(1)} - \sum_i \int_{t_1}^{t_2} dt \frac{d}{dt}\left.\frac{\partial L}{\partial \dot{q}_i}\right|_{q^c(t)} \delta q_i(t) \tag{6}$$

Substituting (6) in (5) and (2)

$$0 = \delta S\left[\delta q_i(t)\right] = \sum_i \int_{t_1}^{t_2} dt \left\{\left.\frac{\partial L}{\partial q_i}\right|_{q^c(t)} - \frac{d}{dt}\left.\frac{\partial L}{\partial \dot{q}_i}\right|_{q^c(t)}\right\} \delta q_i(t) \tag{7}$$

Which as $\delta q_i(t)$ can be made arbitrary at any time instant between t_1 and t_2 it follows that the classical trajectory has to satisfy the Euler-Lagrange equation,

$$\left.\frac{\partial L}{\partial q_i}\right|_{q^c(t)} - \frac{d}{dt}\left.\frac{\partial L}{\partial \dot{q}_i}\right|_{q^c(t)} = 0 \tag{8}$$

II. CONSERVATION LAWS

It is important to realize three points: (1) the very definition of the quantities such as momentum, angular momentum, etc., depends on the conservation laws. Even in the cases when these quantities are not conserved their working definition is inherited from the setting where they are conserved. (2) Classically the conservation laws follow from the continuous symmetries as we will see. This is the essence of the Noether theorem.

Let us define what we mean by symmetry in classical mechanics. Consider any classical trajectory, $q_i^c(t)$ which locally (and globally but that is not important now) minimizes the action. Now the statement is that for a symmetry operation $\hat{O}$ to be a symmetry operation the trajectory $\hat{O}q_i^c$ is also a solution of the same equations of motion. The examples of symmetry operations are given below. We can reformulate this property such that it would remind the quantum mechanical definition more closely. To this end recall that the classical trajectory is determined by the initial condition at a given time t_1, say. Given $q_i(t_1), \dot{q}_i(t_1)$ the second order EL equation (8) fixes the trajectory at all subsequent times, $t > t_1$. The resulting trajectory can be written as $q_i(t) = \hat{U}(t, t_1)[q_i(t_1), \dot{q}_i(t_1)]$ where the operator $\hat{U}(t, t_1)$ is closely analogous to the quantum mechanical propagator. The operation $\hat{O}$ may be stated to be a symmetry operation provided it satisfies

$$\hat{U}(t, t_1)[\hat{O}q_i(t_1), \hat{O}\dot{q}_i(t_1)] = \hat{O}\left\{\hat{U}(t, t_1)[q_i(t_1), \dot{q}_i(t_1)]\right\}, \tag{9}$$

which can be said as to mean the commutation relation between the symmetry operation and propagator. In other words if we "rotate" the initial condition the whole trajectory rotates accordingly while remaining a solution of (8).

The general way of deriving the conservation laws out of a continuous symmetry is as follows. When we apply a symmetry transformation to a given trajectory that satisfies (8) we should get at most a boundary term addition to the action,

$$\int_{t_1}^{t_2} dt L(q'_i(t), \dot{q}'_i(t), t) = F(q_i(t_2), t_2) - F(q_i(t_2), t_2) + \int_{t_1}^{t_2} dt L(q_i(t), \dot{q}_i(t), t), \tag{10}$$

where $q'_i(t)$ is a transformed tarjectory. Indeed, the presence of the boundary terms in (10) will not change the fact that the transformed trajectory also minimizes the action. In most cases I consider the boundary term will be absent.

Equation (10) is general. Now let us consider the classical trajectories, and focus on infinitesimal symmetry operations. Note that we need the symmetry to be a continuous one in order to be able to consider the infinitesimal transformations. Parity for instance cannot be made by tiny steps. (Nevertheless quantum mechanically with the parity symmetry we associate parity conservation when the parity commutes with the Hamiltonian).

For the infinitesimal transformations the difference between the two integrals appearing on the left and right hand sides of (10) would be zero if not for the shift of the coordinates and/or time derivatives of the coordinates at the end times t_1 and t_2. (It becomes a bit more involved when the times t_1 and t_2 change, see below). It then follows that we have a "conservation" law as something at the time t_2 must be equal to something at the time t_1.

A. Momentum conservation

The momentum conservation is associated with the translational symmetry,

$$\hat{O}\boldsymbol{r}_i = \boldsymbol{r}_i + \boldsymbol{\delta}, \tag{11}$$

where $\boldsymbol{r}_i$ is the position radius vector of a particle number i. In terms of trajectories, $\hat{O}\boldsymbol{r}_i(t) = \boldsymbol{r}_i(t) + \boldsymbol{\delta}$, and as a result, $\hat{O}\dot{\boldsymbol{r}}_i(t) = \dot{\boldsymbol{r}}_i(t)$. We have as advertized in (10),

$$\int_{t_1}^{t_2} dt L(\boldsymbol{r}_i(t) + \boldsymbol{\delta}, \dot{\boldsymbol{r}}_i(t), t) = \int_{t_1}^{t_2} dt L(\boldsymbol{r}_i(t), \dot{\boldsymbol{r}}_i(t), t) \tag{12}$$

Make an expansion in infinitesimal (and constant in time and space!) vector $\boldsymbol{\delta}$ to get,

$$\begin{aligned}
0 &= \int_{t_1}^{t_2} dt L(\boldsymbol{r}_i(t) + \boldsymbol{\delta}, \dot{\boldsymbol{r}}_i(t), t) - \int_{t_1}^{t_2} dt L(\boldsymbol{r}_i(t), \dot{\boldsymbol{r}}_i(t), t) \approx \sum_i \int_{t_1}^{t_2} dt \left\{ \frac{\partial L(\boldsymbol{r}_i(t), \dot{\boldsymbol{r}}_i(t), t)}{\partial \boldsymbol{r}_i} \cdot \boldsymbol{\delta} + \frac{\partial L(\boldsymbol{r}_i(t), \dot{\boldsymbol{r}}_i(t), t)}{\partial \dot{\boldsymbol{r}}_i} \cdot \dot{\boldsymbol{\delta}} \right\} \\
&= \sum_i \int_{t_1}^{t_2} dt \left\{ \frac{\partial L(\boldsymbol{r}_i(t), \dot{\boldsymbol{r}}_i(t), t)}{\partial \boldsymbol{r}_i} \cdot \boldsymbol{\delta} + \frac{d}{dt}\left[\frac{\partial L(\boldsymbol{r}_i(t), \dot{\boldsymbol{r}}_i(t), t)}{\partial \dot{\boldsymbol{r}}_i} \cdot \boldsymbol{\delta} \right] - \frac{d}{dt}\left[\frac{\partial L(\boldsymbol{r}_i(t), \dot{\boldsymbol{r}}_i(t), t)}{\partial \dot{\boldsymbol{r}}_i} \right] \cdot \boldsymbol{\delta} \right\} \\
&\overset{(8)}{=} \sum_i \int_{t_1}^{t_2} dt \left\{ \frac{d}{dt}\left[\frac{\partial L(\boldsymbol{r}_i(t), \dot{\boldsymbol{r}}_i(t), t)}{\partial \dot{\boldsymbol{r}}_i} \cdot \boldsymbol{\delta} \right] \right\} = \sum_i \boldsymbol{p}_i(t_2) - \sum_i \boldsymbol{p}_i(t_1), \qquad \boldsymbol{p}_i = \frac{\partial L(\boldsymbol{r}_i(t), \dot{\boldsymbol{r}}_i(t), t)}{\partial \dot{\boldsymbol{r}}_i}
\end{aligned} \tag{13}$$

which serves as the definition of the linear momentum.

B. Conservation of angular momentum

Consider the overall infinitesimal rotation,

$$\hat{O}\boldsymbol{r}_i = \boldsymbol{r}_i + d\phi\hat{n}\times\boldsymbol{r}_i, \quad \hat{O}\dot{\boldsymbol{r}}_i = \dot{\boldsymbol{r}}_i + d\phi\hat{n}\times\dot{\boldsymbol{r}}_i, \tag{14}$$

Let's repeat the arguments, as before,

$$\int_{t_1}^{t_2} dt L(\boldsymbol{r}_i(t) + d\phi\hat{n}\times\boldsymbol{r}_i, \dot{\boldsymbol{r}}_i(t) + d\phi\hat{n}\times\dot{\boldsymbol{r}}_i, t) = \int_{t_1}^{t_2} dt L(\boldsymbol{r}_i(t), \dot{\boldsymbol{r}}_i(t), t) \tag{15}$$

Make an expansion,

$$\begin{aligned}
0 =& \int_{t_1}^{t_2} dt L(\boldsymbol{r}_i(t) + d\phi\hat{n}\times\boldsymbol{r}_i, \dot{\boldsymbol{r}}_i(t) + d\phi\hat{n}\times\dot{\boldsymbol{r}}_i, t) - \int_{t_1}^{t_2} dt L(\boldsymbol{r}_i(t), \dot{\boldsymbol{r}}_i(t), t) \\
\approx& \sum_i \int_{t_1}^{t_2} dt \left\{ \frac{\partial L(\boldsymbol{r}_i(t), \dot{\boldsymbol{r}}_i(t), t)}{\partial \boldsymbol{r}_i} \cdot d\phi\hat{n}\times\boldsymbol{r}_i + \frac{\partial L(\boldsymbol{r}_i(t), \dot{\boldsymbol{r}}_i(t), t)}{\partial \dot{\boldsymbol{r}}_i} \cdot d\phi\hat{n}\times\dot{\boldsymbol{r}}_i \right\} \\
=& d\phi\hat{n}\cdot \sum_i \int_{t_1}^{t_2} dt \left\{ \boldsymbol{r}_i \times \frac{\partial L(\boldsymbol{r}_i(t), \dot{\boldsymbol{r}}_i(t), t)}{\partial \boldsymbol{r}_i} + \dot{\boldsymbol{r}}_i \times \frac{\partial L(\boldsymbol{r}_i(t), \dot{\boldsymbol{r}}_i(t), t)}{\partial \dot{\boldsymbol{r}}_i} \right\} \\
=& d\phi\hat{n}\cdot \sum_i \int_{t_1}^{t_2} dt \left\{ \boldsymbol{r}_i \times \frac{\partial L(\boldsymbol{r}_i(t), \dot{\boldsymbol{r}}_i(t), t)}{\partial \boldsymbol{r}_i} + \frac{d}{dt}\left[\boldsymbol{r}_i \times \frac{\partial L(\boldsymbol{r}_i(t), \dot{\boldsymbol{r}}_i(t), t)}{\partial \dot{\boldsymbol{r}}_i} \right] - \boldsymbol{r}_i \times \frac{d}{dt}\left[\frac{\partial L(\boldsymbol{r}_i(t), \dot{\boldsymbol{r}}_i(t), t)}{\partial \dot{\boldsymbol{r}}_i} \right] \right\} \\
\overset{(8)}{=}& d\phi\hat{n}\cdot \left[\sum_i \boldsymbol{L}_i(t_2) - \sum_i \boldsymbol{L}_i(t_1) \right], \qquad \boldsymbol{L}_i = \boldsymbol{r}_i \times \boldsymbol{p}_i
\end{aligned} \tag{16}$$

Note that the invariance wrt to rotations around axis $\hat{n}$ gives the conservation of the component, $\hat{n}\cdot\sum_i \boldsymbol{L}_i$ but not other components.

C. Energy Conservation

Given the solution $q_i^c(t)$ the trajectories $q_i^c(t-\delta t)$ shifted in time is also a solution provided the EL equations (8) do not contain time explicitly. This will happen if the Lagrangian will not contain such explicit time dependence. Let's try to extract the conservation law out of this seemingly trivial symmetry. We have then,

$$\int_{t_1+\delta}^{t_2+\delta} dt L(q_i(t-\delta), \dot{q}_i(t-\delta)) = \int_{t_1}^{t_2} dt L(q_i(t), \dot{q}_i(t)) \tag{17}$$

And again following the same logic,

$$\begin{aligned}
0 =& \int_{t_1+\delta}^{t_2+\delta} dt L(q_i(t-\delta), \dot{q}_i(t-\delta)) - \int_{t_1}^{t_2} dt L(q_i(t), \dot{q}_i(t)) \\
\approx& \delta\left[L(q_i(t_2), \dot{q}_i(t_2)) - L(q_i(t_1), \dot{q}_i(t_1)) \right] + \int_{t_1}^{t_2} dt \left\{ L(q_i(t-\delta), \dot{q}_i(t-\delta)) - L(q_i(t), \dot{q}_i(t)) \right\}
\end{aligned} \tag{18}$$

The last term again reads,

$$\begin{aligned}
&\int_{t_1}^{t_2} dt \left\{ L(q_i(t-\delta), \dot{q}_i(t-\delta)) - L(q_i(t), \dot{q}_i(t)) \right\} \approx -\delta \sum_i \int_{t_1}^{t_2} dt \left\{ \frac{\partial L}{\partial q_i}\dot{q}_i + \frac{\partial L}{\partial \dot{q}_i}\ddot{q}_i \right\} \\
&= -\delta \sum_i \int_{t_1}^{t_2} dt \left\{ \frac{\partial L}{\partial q_i}\dot{q}_i + \frac{d}{dt}\left[\frac{\partial L}{\partial \dot{q}_i}\dot{q}_i \right] - \frac{d}{dt}\left[\frac{\partial L}{\partial \dot{q}_i} \right]\dot{q}_i \right\} \overset{(8)}{=} -\delta \sum_i \int_{t_1}^{t_2} dt \frac{d}{dt}\left[\frac{\partial L}{\partial \dot{q}_i}\dot{q}_i \right] = -\delta \left[\sum_i p_i\dot{q}_i \Big|_{t=t_2} - \sum_i p_i\dot{q}_i \Big|_{t=t_1} \right].
\end{aligned} \tag{19}$$

Collecting (18) (27) we get the energy conservation law,

$$\delta\left\{\left[L-\sum_i p_i\dot{q}_i\right]\Big|_{t_2}-\left[L-\sum_i p_i\dot{q}_i\right]\Big|_{t_1}\right\} \tag{20}$$

and correspondingly the definition of energy reads,

$$E=-L+\sum_i p_i\dot{q}_i \tag{21}$$

One might wonder where the explicit time independence of the Lagrangian comes in. We could write an identity that parallels (17) as follows,

$$\int_{t_1+\delta}^{t_2+\delta} dtL(q_i(t-\delta),\dot{q}_i(t-\delta),t-\delta)=\int_{t_1}^{t_2} dtL(q_i(t),\dot{q}_i(t),t) \tag{22}$$

which is always true. But in this case the expansion to the linear order in δ would include the term,

$$-\delta\int_{t_1}^{t_2} dt\frac{\partial L}{\partial t} \tag{23}$$

which does not amount to some boundary term because the derivative appearing in (23) is the partial and not the total one. Of course this obstacle does not appear in the case without the explicit time dependence because in this case we really have a variation of the trajectory: the shift in time sits entirely in the arguments of the coordinates and the velocities. For this reason EL equations, (8) guarantee that the variation of the action due to the time shift will produce the boundary terms only.

In this respect the conservation law (28) requires similar assumptions to the other two conservation laws that we have considered so far, (13) and (16). Indeed consider the collection of (possibly interacting via potential forces) particles put into the external potential, $V_{ext}(\boldsymbol{r})$ that adds the term $-\sum_i V_{ext}(\boldsymbol{r}_i)$ to the Lagrangian. We still could consider the shift transformation, (11) that eventually lead us to the conservation laws. Similarly we could write down a trivial identity equation analogous to (23). But, and that is important, with the shifted potential $V'_{ext}(\boldsymbol{r})=V_{ext}(\boldsymbol{r}-\boldsymbol{\delta})$. Without taking care to shift the potential we would not have the same action after transformation. The replacement of $V_{ext}(\boldsymbol{r})$ by $V_{ext}(\boldsymbol{r}-\boldsymbol{\delta})$ adds to the action the extra piece $\sum_i\int_{t_1}^{t_2} dt[V_{ext}(\boldsymbol{r}_i)-V_{ext}(\boldsymbol{r}_i-\boldsymbol{\delta})]$ that gives a contribution that does not come from the variation of trajectories and will prevent us form writing it as a boundary term.

Of course, one may decide to leave the potential to avoid the offending extra pieces, as is but then the action will not stay the same, and the starting (13) will not hold.

In conclusion if the Lagrangian is invariant under some continuous family of transformations one can consider the infinitesimal transformation that affects the coordinates (trajectory). Then on the trajectories satisfying EL (8) this zero change means that the two boundary terms (at t_1 and t_2) must cancel, or in other words some quantity which depends on transformation must be the same at t_2 and t_1. The conclusion does not change if as a result of transformation a total time derivative of any function of coordinates and time is added to the Lagrangian. In this case this function would be added to the expression for the conserved quantity.

III. RELATIONSHIP BETWEEN THE CLASSICAL ACTION AND MOMENTUM AND ENERGY

We now consider the classical action as a function of the initial time and coordinates and final time and coordinates. We define

$$S[(t_1,q_{j,1});(t_2,q_{j,2})]=\min_{q(t)}\int_{t_1}^{t_2} dtL(q_j,\dot{q}_j,t) \tag{24}$$

where the minimization is over the trajectories passing subject to the constrain $q_j(t_1)=q_{j,1}$ and $q_j(t_2)=q_{j,2}$. In other words the above definition is the action on the actual trajectories satisfying the above constrain.

A. Relation to the momentum

Let's compute the spatial and temporal partial derivatives of the classical action (24). Namely, we change the ith final coordinate $q_{i,2}\to q_{i,2}+\delta q_i$ and the action changes as the trajectory must adjust to the change of the final point.

So we have the change in final position $\delta q_{j,2} = \delta_{i,j}\delta q_{i,2}$

$$\frac{\partial S[(t_1, q_{j,1}); (t_2, q_{j,2})]}{\partial q_{i,2}}\delta q_{i,2} = \int_{t_1}^{t_2} dt\, [L(q_j(t) + \delta q_j(t), \dot{q}_j(t) + \delta\dot{q}_j(t)) - L(q_j(t), \dot{q}_j(t))]$$
$$= \int_{t_1}^{t_2} dt \sum_j [(\partial L/\partial q_j)\delta q_j + (\partial L/\partial \dot{q}_j)\delta\dot{q}_j = \int_{t_1}^{t_2} dt \sum_j \left[(\partial L/\partial q_j)\delta q_j + \frac{d}{dt}[(\partial L/\partial \dot{q}_j)\delta q_j] - \frac{d}{dt}[(\partial L/\partial \dot{q}_j)]\delta \dot{q}_j\right]$$
$$= \sum_j (\partial L/\partial \dot{q}_j)\Big|_{t=t_2, q_j=q_{j,2}} \delta q_{j,2} - \sum_j (\partial L/\partial \dot{q}_j)\Big|_{t=t_1, q_j=q_{j,1}} \delta q_{j,1} = p_i \delta q_{i,2} \tag{25}$$

And we obtain the important relation,

$$p_{i,2} = \frac{\partial S[(t_1, q_{j,1}); (t_2, q_{j,2})]}{\partial q_{i,2}} \tag{26}$$

So the momentum is the derivative of the action with the end point of the trajectory.

B. Relation to the energy

Consider the trajectory specified as above by the initial and final times and coordinates at these times. Now consider the same trajectory traversed by the system during the time interval , $[t_1, t_2]$ in the interval $[t_1, t_2 + dt]$. What we mean by that is the particles are just allowed to move for another infinitesimal time interval, $[t_2, t_2 + dt]$ such that the distance they cover is determined by the their velocities at time t_2. Namely the trajectory we had for $[t_1, t_2]$ got extended by adding a small piece the particles cover during the interval $[t_2, t_2 + dt]$. This means that on the extended trajectory we have the change in the positions of the end points as follows, $dq_{2,j} = \dot{q}_j(t_2)dt$. Because the new trajectory is just an extension of the old one we have the change in the action given by

$$dS = L(q_j, \dot{q}_j)\big|_{t=t_2} dt \tag{27}$$

On the other hand we have for the total differential,

$$dS = \frac{\partial S}{\partial t}\Big|_{t=t_2} dt + \sum_j \frac{\partial S}{\partial q_j}\Big|_{t=t_2} dq_j = \frac{\partial S}{\partial t}\Big|_{t=t_2} dt + \sum_j \frac{\partial S}{\partial q_j}\Big|_{t=t_2} \dot{q}_j dt \tag{28}$$

Comparing (27) and (28) and using (26) we get finally,

$$\frac{\partial S}{\partial t} = L - \sum_j p_j \dot{q}_j = -E \tag{29}$$

IV. RELATIVISTIC ENERGY AND MOMENTUM

Relativistically invariant free particle Lagrangian reads

$$L(\boldsymbol{x}, \dot{\boldsymbol{x}}) = -mc^2\sqrt{1 - \dot{\boldsymbol{x}}^2/c^2} \tag{30}$$

The solution of the classical equation of motion simply that the velocity is constant, $\dot{\boldsymbol{x}} = \boldsymbol{v}$. This follows as the Lagrangian does not depend on the coordinate $\boldsymbol{x}$. We wish now to find the momentum and energy of a particle propagating with this velocity $\boldsymbol{v}$. Consider the trajectory (straight line) passing through $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$ at times t_1 and t_2 respectively. Lets compute the action on actual (the straight line) trajectory,

$$S(\boldsymbol{x}_1, t_1; \boldsymbol{x}_2, t_2) = -mc^2(t_2 - t_1)\sqrt{1 - \frac{(\boldsymbol{x}_1 - \boldsymbol{x}_2)^2}{(t_1 - t_2)^2 c^2}} = -mc\sqrt{(t_1 - t_2)^2 c^2 - (\boldsymbol{x}_1 - \boldsymbol{x}_2)^2}\,. \tag{31}$$

Note that this action is manifestly Lorentz invariant. Note also that this is true for any trajectory by construction. Then according to (26)

$$\boldsymbol{p} = \frac{\partial S}{\partial \boldsymbol{x}_2} = -mc\frac{-(\boldsymbol{x}_2 - \boldsymbol{x}_1)}{\sqrt{(t_1 - t_2)^2 c^2 - (\boldsymbol{x}_1 - \boldsymbol{x}_2)^2}} = \frac{m\boldsymbol{v}}{\sqrt{1 - v^2/c^2}} \tag{32}$$

$$E = -\frac{\partial S}{\partial t_2} = mc\frac{c^2(t_2 - t_1)}{\sqrt{(t_1 - t_2)^2 c^2 - (\boldsymbol{x}_1 - \boldsymbol{x}_2)^2}} = \frac{mc^2}{\sqrt{1 - v^2/c^2}} \tag{33}$$

The action is a scalar by construction. Furthermore the set of four $(ct, -x, -y, -z) = (x_0, x_1, x_2, x_3)$ are the covariant components of the coordinates 4-vector. Therefore the four quantities $(E/c, p_x, p_y, p_z) = (p^0, p^1, p^2, p^3)$ are four contravariant components of the four-momentum vector. Indeed, we have from above,

$$p^\mu = -\frac{\partial S}{\partial x_\mu} \tag{34}$$