

## LL § 8 Least Action Principle

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the actual trajectory is such that  
the action  $S$  is minimal, and as  
a result the variation vanishes,  $\delta S = 0$

Let's find the action for a free particle.

Note: the action is Lorentz invariant.

This has to be so to ensure that all observers  
agree on the trajectory.

The action must be integral of a differential along  
a trajectory  $\Rightarrow$

$$S = -\alpha \int_a^b ds \quad (15)$$

where  $\alpha$  is a constant,  $ds^2 = c^2 dt^2 - (\vec{dx})^2$  an interval  
a and b are the initial and final events  
and the integral is along a world line with  
fixed a and b that occurred at times  $t_1, t_2$ .

On a given trajectory  $\vec{x}(t)$ ,  $ds^2 = c^2 dt^2 - v^2(t) dt^2$   
 $\Rightarrow ds = c dt \sqrt{1 - v^2(t)/c^2}$

$$S = \int_{t_1}^{t_2} dt L, \text{ where the Lagrangean} \quad (16)$$

$$L = -\alpha c \sqrt{1 - v^2(t)/c^2} \quad (17)$$

Let's fix a constant  $\alpha$ .

Imagine that the trajectory of a particle has  
to pass through  $(t_1, x_1)$  and  $(t_2, x_2)$  (initial and

final points are fixed]. We certainly have to assume that the interval  $dS_{12}^2 = c^2(t_1 - t_2)^2 - (\vec{x}_1 - \vec{x}_2)^2 > 0$  (is time-like). In the system frame  $K'$  where  $\vec{x}'_1 = \vec{x}'_2$  with some  $t'_1$  and  $t'_2$ ,  $t'_1 \neq t'_2$  the maximal integral  $\int_a^{b'} dS$  is achieved for the trivial trajectory in  $K'$ , where the particle stays at rest at  $\vec{x}'_1 = \vec{x}'_2$ . Such a trajectory is compatible with the boundary conditions because  $\vec{x}'_1 = \vec{x}'_2$  in  $K'$ .

Since we are looking at the least action principle the constant  $\alpha$  in (15) is positive.

To fix its magnitude go to the non-relativistic limit,  $v/c \ll 1$ ,

$$L = -\alpha c \sqrt{1 - v^2/c^2} \approx -\alpha c + \frac{\alpha v^2}{2c}$$

as in non-relativistic limit,  $L = mv^2/2$  and a constant is irrelevant (it does not enter equations of motion), we must have  $\boxed{\alpha = mc}$

$$\Rightarrow S = -mc \int_a^b ds , \quad (18)$$

$$S = \int_{t_1}^{t_2} dt L , \quad L = -mc^2 \sqrt{1 - v^2/c^2} \quad (19)$$

## §9 Energy and Momentum

$$\vec{p} = \frac{\partial L}{\partial \vec{v}} , \text{ with (19)} \quad \vec{p} = \frac{m\vec{v}}{\sqrt{1 - v^2/c^2}}$$

## Energy

$$E = \vec{p} \cdot \vec{v} - L = \frac{mv^2}{\sqrt{1-v^2/c^2}} + mc^2\sqrt{1-v^2/c^2} \Rightarrow$$

$$E = \frac{mc^2}{\sqrt{1-v^2/c^2}}$$

$E(v=0) = mc^2$  is rest energy

$$E(v \ll c) = mc^2 + \frac{mv^2}{2}$$

The mass of a composite body differs from the sum of masses of constituents as is the case in nuclear reactions: the mass deficit is the energy released in the reaction.

$$\frac{E^2}{c^2} - \vec{p}^2 = \frac{m^2 c^2}{1-v^2/c^2} - \frac{m^2 v^2}{1-v^2/c^2} = m^2 c^2$$

$$\Rightarrow E^2 = c^2(p^2 + m^2 c^2) \quad (20)$$

$$\Rightarrow \text{Hamiltonian} \quad \boxed{H = c \sqrt{\vec{p}^2 + m^2 c^2}} \quad (21)$$

$$\text{Another relation} \quad \vec{p} = \frac{E \vec{v}}{c^2}$$

At  $m > 0 \quad v < c$  as otherwise  $E, p = \infty$

But for  $m=0$ , from (20)  $E = cp$

Let's rederive the above relations in covariant notations. Least action principle gives

$$\delta S = -mc \delta \int_a^b ds = 0$$

$$ds = \sqrt{dx_i dx^i}$$

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$$\delta ds = \frac{dx_i \delta x^i}{ds} \Rightarrow \delta s = -mc \int_a^b \frac{dx_i \delta x^i}{ds}$$

We've introduced previously the 4-velocity

$$(U^0, \vec{U}) = (\gamma_u c, \gamma_u \vec{u}) = \left( \frac{dx^0}{dt}, \frac{d\vec{x}}{dt} \right)$$

We now introduce notation  $u^i$  for a dimensionless 4-velocity  $(u^0, \vec{u}) = \left( \frac{dx^0}{ds}, \frac{d\vec{x}}{ds} \right)$  following L.L.

as  $ds = c dt$  we clearly have

$$(u^0, \vec{u}) = c^{-1} (U^0, \vec{U}) = (\gamma_u, \gamma_u \vec{u}/c)$$

$\downarrow$  spatial components  $\neq$  regular velocity !  
of 4-velocity  $u^i$

$$\delta s = -mc \int_a^b u_i \delta x^i . \text{ Integration by parts gives}$$

$$\delta s = -mc u_i \delta x^i \Big|_a^b + mc \int_a^b \delta x^i \frac{du_i}{ds} ds$$

$$\text{Equation of motion : } \frac{du_i}{ds} = 0 , \Rightarrow$$

4-velocity is constant. Note that although by itself it is obvious, nevertheless it is written in Lorentz covariant form as should be the case for all equations encountered in the theory.

On a classical trajectories

$$\delta s = -mc u_i \delta x^i \Big|_a^b$$

(22)

From mechanics, momentum  $\vec{p} = \frac{\partial S}{\partial \vec{x}}$ ,  $E = -\frac{\partial S}{\partial t}$  (p.5)

where the action is computed along the true trajectories and is considered as a function of an end point,  $(t, \vec{x})$ . Define a 4-momentum

$$p_i = -\frac{\partial S}{\partial x^i}, \text{ then } p_0 = -\frac{\partial S}{c \partial t} = \bar{c}E$$

$$p_i = -\frac{\partial S}{\partial x^i} = -(\vec{p})_i$$

By raising its index we get

$$p^i = (p^0, \vec{p}) = (\bar{c}E, \vec{p}) = \left(-\frac{\partial S}{c \partial t}, \frac{\partial S}{\partial \vec{x}}\right)$$

consistently with the definition in mechanics.

From (22)

$$p^i = mc u^i, \text{ i.e. } \left(\frac{E}{c}, \vec{p}\right) \text{ is a 4-vector.}$$

Since

$$u^i u_i = \gamma_u^2 - \gamma_u^2 \beta_u^2 = 1$$

$p^i p_i = m^2 c^2$  is constant and invariant

As for any other 4-vector it transforms as

$$p_x = \frac{p_x' + \frac{v}{c} \frac{E'}{c}}{\sqrt{1 - v^2/c^2}}, \quad \frac{E'}{c} = \frac{\frac{E}{c} + \frac{v}{c} p_x'}{\sqrt{1 - v^2/c^2}}$$

$$p_y = p_y', \quad p_z = p_z'.$$

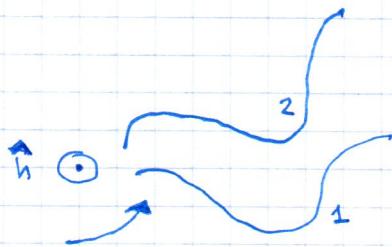
## § 14 L.L. Angular Momentum

Let's get back to non-relativistic domain, where the angular momentum

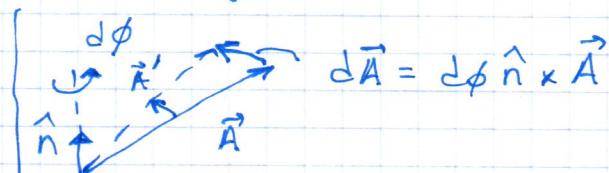
$$\hat{M} = \sum [ \vec{r} \times \vec{p} ]$$

is conserved for an isolated system. Here the sum runs over all the particles.

Let's derive it from Least Action Principle. Imagine the system is invariant with respect to rotations relative to the given axis  $\hat{n}$



Here the action on trajectory 2 is the same as action on trajectory 1, as 2 is obtained by rotating 1 relative to  $\hat{n}$ .



Any vector  $\vec{A}$  is transformed into a rotated vector  $\vec{A}'$  such that

$$d\vec{A} = \vec{A}' - \vec{A} = d\phi \hat{n} \times \vec{A}$$

For arbitrary variation of a trajectory  $\delta \vec{q}_i(t)$

$$\delta S = \delta \int_{t_1}^{t_2} dt L(\vec{q}_i, \dot{\vec{q}}_i, t) = \int_{t_1}^{t_2} dt \left( \frac{\partial L}{\partial \vec{q}_i} \delta \vec{q}_i + \frac{\partial L}{\partial \dot{\vec{q}}_i} \delta \dot{\vec{q}}_i \right) =$$

$$= \sum_i \left[ \frac{\partial L}{\partial \dot{\vec{q}}_i} \delta \vec{q}_i \right]_{t_1}^{t_2} + \sum_i \int_{t_1}^{t_2} dt \left( \frac{\partial L}{\partial \vec{q}_i} - \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{\vec{q}}_i} \right) \right) \delta \vec{q}_i$$

$$= \sum_i \left[ \frac{\partial L}{\partial \dot{\vec{q}}_i} \delta \vec{q}_i \right]_{t_1}^{t_2}, \text{ since on trajectories satisfying equation of motion the second term drops.}$$

Consider as  $\delta \vec{q}_i$  and  $\dot{\delta \vec{q}}_i$  rotation of the given trajectory  $\vec{q}_i(t)$ ,  $\dot{\vec{q}}_i(t)$ . Then from symmetry  $\delta S = 0$ :

$$0 = \sum_i \left[ \frac{\partial L}{\partial \dot{\vec{q}}_i} \delta \vec{q}_i \right]_{t_1}^{t_2}$$

On the other hand,  $\delta \vec{q}_i(t) = d\phi \hat{n} \times \vec{q}_i(t)$  for all  $t$ . and by definition  $\vec{p}_i = \frac{\partial L}{\partial \dot{\vec{q}}_i}$ . Therefore

$$0 = \sum_i \vec{p}_i \cdot d\phi (\hat{n} \times \vec{q}_i) \Big|_{t_1}^{t_2} \Rightarrow \hat{n} \cdot \left( \sum_i \vec{q}_i \times \vec{p}_i \right) \Big|_{t=t_2} = \hat{n} \cdot \left( \sum_i \vec{q}_i \times \vec{p}_i \right) \Big|_{t=t_1}$$

$\Rightarrow$  conservation of  $\hat{n}$ -th component of angular momentum.

Now let's get back to relativistic domain.

Let's make an infinitesimal rotation

$$x^i \rightarrow x^{i'} \text{ such that } x_i x^i = x'_i x'^i$$

$$x'^i - x^i = x_k \delta \varSigma^{ik}$$

$$(x^i + x_k \delta \varSigma^{ik})(x_i + x^{k'} \delta \varSigma_{ik'}) = x^i x_i$$

$$x_k x_i \delta \varSigma^{ik} + x^i x^{k'} \delta \varSigma_{ik'} = 0$$

$$x^i x^k \delta \varSigma_{ik} = 0 \Rightarrow \text{holds for any } x^i \Rightarrow$$

$\Rightarrow \delta \varSigma_{ik}$  is antisymmetric :  $\delta \varSigma_{ki} = -\delta \varSigma_{ik}$

$$\text{We've seen } \delta S = - \sum p^i \delta x_i \Big|_2$$

(note : “-” sign as compared to the non-relativistic expressions is obtained as the space part of a metric is negative :  $g_{00}=1$ ,  $g_{11}=g_{22}=g_{33}=-1$ )

With  $\delta x_i = \delta \Omega_{ik} x^k$  and since the rotation leaves action invariant,

(p.8)

$$O = \delta S = -\delta \Omega_{ik} \sum p^i x^k \Big|_a^b \stackrel{\delta \Omega_{ik} = -\delta \Omega_{ki}}{=} -\delta \Omega_{ik} \frac{1}{2} \sum (p^i x^k - p^k x^i) \Big|_a^b$$

$\Rightarrow$  6 conservation laws:

$$M^{ik} = \sum (x^i p^k - x^k p^i) \text{ are conserved}$$

$M^{ik}$  is (anti-symmetric) angular momentum tensor.

The spatial components are the components of a non-relativistic angular momentum

$$M^{23} = \sum (y p_z - z p_y) \equiv M_x$$

$$-M^{13} = \sum (-x p_z + z p_x) \equiv M_y$$

$$M^{12} = \sum (x p_y - y p_x) \equiv M_z$$

But we get another triad of conserved quantities that transforms as 3vector under rotations

$$(M^{01}, M^{02}, M^{03}) = c \sum (t \vec{p} - E \vec{r}/c) \text{ is conserved}$$

$$\Rightarrow \sum (t \vec{p} - \frac{E \vec{r}}{c}) = \text{const}$$

$\sum E$  is conserved as well  $\Rightarrow$

$$\frac{\sum E \vec{r}}{\sum E} - t \frac{c^2 \sum \vec{p}}{\sum E} = \text{const}$$

Define relativistic center of mass

$$\vec{R} = \frac{\sum E \vec{r}}{\sum E}$$

(p.9)

The conservation Law tells us that the point  $\vec{R}$  above moves with a constant velocity

$$\vec{V} = \frac{c^2 \sum \vec{p}}{\sum E}$$

For a single particle  $\vec{p} = \frac{E \vec{v}}{c^2}$ , and therefore

$\vec{V}$  defines the velocity of motion as a whole with the energy and momentum equal to the total energy and total momentum.

In the limit  $v \ll c$   $R \rightarrow \frac{\sum m \vec{r}}{\sum m}$  is a usual center of mass.

NOTE: The three components of  $\vec{R}$  are not spatial components of any 4-vector! Therefore the point  $\vec{R}$  as defined above is different for different observers.