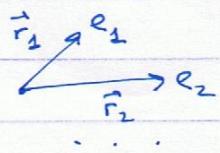


P.28 L1. Four-dimensional vector of current

Charge density $\rho(\vec{x}, t)$ is defined such that
 $\rho(\vec{x}, t) dV$ is the charge contained in the volume dV

$$\int_V \rho(\vec{x}, t) dV = \text{total charge in the volume } V \quad (1)$$

 For the set of discrete charges e_a located at \vec{r}_a

$$\rho(\vec{x}, t) = \sum_a e_a \delta(\vec{x} - \vec{r}_a(t)) \quad (2)$$

Eq. (2) satisfies the requirement (1).

The magnitude of each point-charge, e_a is a scalar by definition, the charge density, however is not a scalar. The product ρdV is invariant as it counts the total charge in a given volume.

Let's introduce 4-current vector

$$de dx^i = \rho dV dx^i = \rho dV dt \frac{dx^i}{dt} \quad (3)$$

 $dx^i = (c dt, \vec{dx} = \vec{v} dt)$

dV contains charge $de = \rho dV$

during the time interval dt , the element dV moves by $d\vec{x} = \vec{v} dt$. The 4 quantities

$dx^i = (c dt, \vec{dx} = \vec{v} dt)$ comprise a 4-vector

$dV dt$ is a scalar since $|\det[\Lambda^\alpha_\beta]| = 1$

$de dx^i$ is a 4-vector.

\Rightarrow Eq.(3) results in the 4-vector of current density

$$\boxed{j^{\mu} = \rho \frac{dx^{\mu}}{dt}}, \quad j^{\mu} = (c\rho, \vec{j}), \text{ where } \vec{j} = \rho \vec{v}$$

because $\frac{dx^{\mu}}{dt} = \frac{1}{c} [cdt, \vec{r}dt] = (c, \vec{v})$

Note: Both ρ and \vec{j} are functions of space-time coordinates, x^{μ} , i.e. $\rho = \rho(t, \vec{x})$, $\vec{j} = j(t, \vec{x})$. We've discussed the meaning of $\rho(t, \vec{x})$; the meaning of $j(t, \vec{x})$ is similar: at each give time t the particles located at \vec{x} have the velocity $\vec{v}(t, \vec{x})$, then $\vec{j}(t, \vec{x}) = \rho(t, \vec{x}) \cdot \vec{v}(t, \vec{x})$.

Note: The total charge in all of the space

$$\int dV \rho = \frac{1}{c} \int dV j^0 = \frac{1}{c} \int dS_{\mu} j^{\mu},$$

where the "surface" element on a surface of $t = \text{const}$ points in time-direction and equals to $dx dy dz \equiv dV$ in magnitude.

$dS_{\mu} = (dS_0, dS_1, dS_2, dS_3)$ can be shown to be a co-variant 4-vector.

Let's rewrite the part of the action describing the matter-field interaction:

$$\begin{aligned} S_{mf} &= - \sum_a \frac{e_a}{c} \int dx^{\mu} A_{\mu} \equiv - \sum_a \frac{e_a}{c} \int_{t_1}^{t_2} dt [c A_a^{\circ}(t, \vec{r}_a(t)) - \\ &- \vec{v}_a(t) \cdot \vec{A}(t, \vec{r}_a(t))] = - \frac{1}{c} \int_{t_A}^{t_B} dt \int d^3x \sum_a e_a \delta(\vec{x} - \vec{r}_a(t)) \times \\ &\times [c A^{\circ}(t, \vec{x}) - \vec{v}(t, \vec{x}) \cdot \vec{A}(t, \vec{x})] = - \frac{1}{c} \int_{x_A^0}^{x_B^0} dx^{\mu} \int d^3x [j^{\mu}(t, \vec{x}) \cdot A_{\mu}(t, \vec{x})] \end{aligned}$$