

§18 Gauge Invariance

$$\vec{E} = -\frac{1}{c} \frac{\partial \vec{A}}{\partial t} - \vec{\nabla} \varphi, \quad \vec{B} = \vec{\nabla} \times \vec{A}$$

are observable that enter the equation of motion:

$$\frac{d\vec{p}}{dt} = e\vec{E} + \frac{e}{c} \vec{v} \times \vec{B}, \quad \frac{dE_{kin}}{dt} = e\vec{E} \cdot \vec{v}$$

The fields \vec{E}, \vec{B} do not change under the gauge transformation

$$A'_k = A_k - \frac{\partial f}{\partial x^k}, \text{ where}$$

$A_k = (\varphi, -\vec{A})$ and f is arbitrary function of space coordinates and time.

This, gauge symmetry is related to the invariance and conservation of charge:

The interaction with the field is described by the action that gives the same equation of motion for A_k and $A_k - \frac{\partial f}{\partial x^k}$,

$$\begin{aligned} -\frac{e}{c} \int_a^b A_i dx^i &\xrightarrow[\text{transformation}]{\text{gauge}} -\frac{e}{c} \int_a^b \left(A_k - \frac{\partial f}{\partial x^k} \right) dx^k = \\ &= -\frac{e}{c} \int_a^b A_k dx^k + \frac{e}{c} \int_a^b \frac{\partial f}{\partial x^k} dx^k \end{aligned}$$

But the additional piece

$$\frac{1}{c} \int_a^b \frac{\partial (ef)}{\partial x^k} dx^k = \frac{e}{c} [f(b) - f(a)]$$

does not affect equations of motion according to the least action principle.

Note that the crucial step was to

replace

$$e \frac{\partial f}{\partial x^k} = \frac{\partial e f}{\partial x^k},$$

i.e. conservation of charge.

In components the gauge transformation

$$\vec{A}' = \vec{A} + \vec{\nabla}f, \quad \varphi' = \varphi - \frac{1}{c} \frac{\partial f}{\partial t}$$

and clearly the fields \vec{E}, \vec{B} are invariant

under gauge transformation

$$\vec{E}' = -\frac{\partial}{c \partial t} (\vec{A} + \vec{\nabla}f) - \vec{\nabla}(\varphi - \frac{1}{c} \frac{\partial f}{\partial t}) = -\frac{\partial \vec{A}}{c \partial t} - \vec{\nabla} \varphi = \vec{E}$$

$$\vec{B}' = \vec{\nabla} \times \vec{A}' = \vec{\nabla} \times \vec{A} = \vec{B}$$

as expected from the previous discussion on
invariance of equations of motion.

Indeed: Equations of motion are written in terms of
fields; \vec{E} and \vec{B} .