Notes on Wave-Equation and Radiation

Contents

I. WAVE-EQUATION IN LORENTZ GAUGE

Starting with the set of Maxwell equations,

$$\nabla \cdot \boldsymbol{D} = \rho, \qquad \nabla \times \boldsymbol{H} = \boldsymbol{J} + \frac{\partial \boldsymbol{D}}{\partial t}$$

$$\nabla \cdot \boldsymbol{B} = 0, \qquad \nabla \times \boldsymbol{E} + \frac{\partial \boldsymbol{B}}{\partial t} = 0 \tag{1}$$

The vector and scalar potentials,

$$\boldsymbol{B} = \nabla \times \boldsymbol{A}, \quad \boldsymbol{E} = -\nabla \Phi - \partial_t \boldsymbol{A} \tag{2}$$

In vacuum, $\boldsymbol{B} = \mu_0 \boldsymbol{H}$, $\boldsymbol{D} = \epsilon_0 \boldsymbol{E}$, and with $c^{-2} = \epsilon_0 \mu_0$, and

$$\nabla \cdot \boldsymbol{E} = \rho/\epsilon_0, \qquad \nabla \times \boldsymbol{B} = \mu_0 \boldsymbol{J} + c^{-2} \frac{\partial \boldsymbol{E}}{\partial t}$$

$$\nabla \cdot \boldsymbol{B} = 0, \qquad \nabla \times \boldsymbol{E} + \frac{\partial \boldsymbol{B}}{\partial t} = 0 \tag{3}$$

$$\nabla \cdot \boldsymbol{E} = \nabla(-\nabla\Phi - \partial_t \boldsymbol{A}) = -\nabla^2 \Phi - \partial_t \nabla \boldsymbol{A} \tag{4}$$

and therefore,

$$\nabla^2 \Phi + \partial_t \nabla \boldsymbol{A} = \rho/\epsilon_0 \tag{5}$$

$$\nabla \times \boldsymbol{B} = \nabla \times (\nabla \times \boldsymbol{A}) = \nabla(\nabla \cdot \boldsymbol{A}) - \nabla^2 \boldsymbol{A} \tag{6}$$

$$\partial_t \boldsymbol{E} = \partial_t(-\nabla\Phi - \partial_t \boldsymbol{A}) = -\nabla \partial_t \Phi - \partial_t^2 \boldsymbol{A} \tag{7}$$

So that

$$\nabla(\nabla \cdot \boldsymbol{A}) - \nabla^2 \boldsymbol{A} = \mu_0 \boldsymbol{J} + c^{-2}(-\nabla \partial_t \Phi - \partial_t^2 \boldsymbol{A}) \tag{8}$$

and the Lorentz gauge condition,

$$\boldsymbol{\nabla} \boldsymbol{A} + c^{-2} \partial_t \Phi = 0 \tag{9}$$

gives rise to a inhomogeneous wave equations,

$$\nabla^2 \Phi - c^{-2} \partial_t^2 \Phi = -\rho/\epsilon_0$$

$$\nabla^2 \boldsymbol{A} - c^{-2} \partial_t^2 \boldsymbol{A} = -\mu_0 \boldsymbol{J} \tag{10}$$

A. Discrete charges as a source

For discrete set of charges labeled by index i,

$$\rho(\boldsymbol{x}, t) = \sum_i e_i \delta(\boldsymbol{x} - \boldsymbol{x}_i(t)), \qquad \boldsymbol{J}(\boldsymbol{x}, t) = \sum_i e_i \delta(\boldsymbol{x} - \boldsymbol{x}_i(t))\dot{\boldsymbol{x}}_i(t). \tag{11}$$

The dipole moment of the discrete charge distribution,

$$\boldsymbol{p}(t) = \int d^3 x' \rho(x', t)\boldsymbol{x}' = \sum_i e_i \boldsymbol{x}_i(t) \tag{12}$$

The useful relation,

$$\int d^3 x \boldsymbol{J}(\boldsymbol{x}, t) = \dot{\boldsymbol{d}}(t) \tag{13}$$

is immediately evident for the case of the set of discrete charges, i.e. definitions, (11). For the general distribution of charges and currents, continuous or discrete it follows from the continuity relation,

$$\dot{\boldsymbol{p}}(t) = \frac{d}{dt} \int d^3 x \boldsymbol{x} \rho(\boldsymbol{x}, t) = \int d^3 x \boldsymbol{x} \frac{\partial \rho(\boldsymbol{x}, t)}{\partial t} = -\int d^3 x \boldsymbol{x} \nabla \cdot \boldsymbol{J}(\boldsymbol{x}, t) \tag{14}$$

which in components tells us

$$[\dot{\boldsymbol{p}}]_l(t) = -\int d^3 x x_l \partial_k J_k(\boldsymbol{x}, t) = -\int d^3 x x_l \partial_k J_k(\boldsymbol{x}, t) = -\int d^3 x \partial_k [x_l J_k(\boldsymbol{x}, t)] + \int d^3 x [\partial_k x_l] J_k(\boldsymbol{x}, t)$$

$$= \int d^3 x \delta_{k,l} J_k(\boldsymbol{x}, t) = \int d^3 x J_l(\boldsymbol{x}, t). \tag{15}$$

B. Energy conservation and Poynting Theorem

Here we review the concept of the energy flux, or Poynting vector. This concept is particularly important in the discussion of radiation. Lets start with the derivation of the Poynting theorem. The change in the mechanical energy, W_{mech}^{LF} due to the fields applying the Lorentz force on charges in a volume V is

$$\frac{dW_{mech}^{LF}}{dt} = \int_V \underbrace{d^3x\rho[\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B}]}_{Force\ on\ d^3x} \cdot \boldsymbol{v} = \int_V d^3x \boldsymbol{E} \cdot \boldsymbol{J} \tag{16}$$

Exclude the current density using the second (Ampere-Maxwell) of (3),

$$\frac{dW_{mech}^{LF}}{dt} = \int_V d^3x \boldsymbol{E} \cdot \frac{1}{\mu_0}\left[\nabla \times \boldsymbol{B} - \epsilon_0 \frac{\partial \boldsymbol{E}}{\partial t}\right] \tag{17}$$

Now write, using Faraday law, fourth of (3)

$$\nabla \cdot (\boldsymbol{E} \times \boldsymbol{B}) = \boldsymbol{B} \cdot (\nabla \times \boldsymbol{E}) - \boldsymbol{E} \cdot (\nabla \times \boldsymbol{B}) = -\boldsymbol{B} \cdot \frac{\partial \boldsymbol{B}}{\partial t} - \boldsymbol{E} \cdot (\nabla \times \boldsymbol{B}) \tag{18}$$

So that,

$$\frac{dW_{mech}^{LF}}{dt} = \int_V d^3x \left[-\frac{1}{\mu_0}\nabla \cdot (\boldsymbol{E} \times \boldsymbol{B}) - \frac{1}{\mu_0}\boldsymbol{B} \cdot \frac{\partial \boldsymbol{B}}{\partial t} - \epsilon_0 \boldsymbol{E} \cdot \frac{\partial \boldsymbol{E}}{\partial t}\right] \tag{19}$$

which then gives the Poynting theorem,

$$\frac{d}{dt}\int_V d^3x \frac{\epsilon_0}{2}\left[\boldsymbol{E} \cdot \boldsymbol{E} + c^2 \boldsymbol{B} \cdot \boldsymbol{B}\right] = -\frac{dW_{mech}^{LF}}{dt} - \frac{1}{\mu_0}\int_V d^3x \nabla \cdot (\boldsymbol{E} \times \boldsymbol{B}) . \tag{20}$$

The energy stored in the EM field,

$$W_{fields} = \int_V d^3x \frac{\epsilon_0}{2}\left[\boldsymbol{E} \cdot \boldsymbol{E} + c^2 \boldsymbol{B} \cdot \boldsymbol{B}\right] \tag{21}$$

Introducing the Poynting vector, or radiation flux vector,

$$\boldsymbol{S} = \boldsymbol{E} \times \boldsymbol{H} = \frac{1}{\mu_0}\boldsymbol{E} \times \boldsymbol{B} \quad (SI), \quad \boldsymbol{S} = \frac{c}{4\pi}\boldsymbol{E} \times \boldsymbol{B} \quad (CGS) \tag{22}$$

we obtain the statement of the total energy, W_{tot} conservation in an **isolated** systems of charges and fields created by these charges,

$$\frac{dW_{tot}}{dt} = \frac{d}{dt}(W_{mech}^{LF} + W_{fields}) = -\oint_{Surf} da \boldsymbol{n} \cdot \boldsymbol{S} \tag{23}$$

which is straightforwardly interpreted as the decrease in the total energy is accounted for by its flow out of the volume V through the surface $Surf$. In the isolated system, the energy transforms from the mechanical to the field energy but the total energy is conserved. In this case when the volume V inflates to infinity and so does the surface $Surf$ enclosing V the right-hand side in (23) should drop to yield the total energy conservation in the whole volume,

$$\frac{d}{dt}(W_{mech}^{LF} + W_{fields}) = 0 \tag{24}$$

1. Steadily radiating systems

In the system radiating steadily the energy is imparted into the system by the external agents. One could think of them as "electricity company" broadly understood. Let us imagine having the periodic driving force and consider the steady state. Then average (23) over a period. Such averaging is denoted by the line put above an averaged quantity,

$$\overline{\frac{d}{dt}(W_{mech}^{LF} + W_{fields})} = -\oint_{Surf} da \boldsymbol{n} \cdot \overline{\boldsymbol{S}} \tag{25}$$

The state of steady radiation is characterized by the condition,

$$\lim_{V \to \infty} \oint_{Surf} da \boldsymbol{n} \cdot \boldsymbol{S} \neq 0 \tag{26}$$

exactly because the total energy in the system, W_{tot} in the whole space is not conserved thanks to the work performed by the "electricity company". On the other hand, in the steady state,

$$\overline{\frac{d}{dt} W_{fields}} = 0 \tag{27}$$

and by definition (16)

$$\int d^3 x \overline{\boldsymbol{E} \cdot \boldsymbol{J}} = - \lim_{Surf \to \infty} \oint_{Surf} da \boldsymbol{n} \cdot \overline{\boldsymbol{S}} \tag{28}$$

The expression on the right hand side of (16) is the average mechanical work done by the field on a radiating system. Clearly, however the total mechanical energy of the system in the steady state when averaged over the period remains constant. The point is that the "electrical company" works against those fields to compensate for their deceleration action on charges in the radiating system. In other words the average power supplied by the "electrical company", P must satisfy, (in steady state!),

$$P + \int d^3 x \overline{\boldsymbol{E} \cdot \boldsymbol{J}} = 0 \tag{29}$$

Combining (28) and (29) we finally arrive at the expression we will normally use,

$$P = \lim_{Surf \to \infty} \oint_{Surf} da \boldsymbol{n} \cdot \overline{\boldsymbol{S}} \,, \tag{30}$$

where it is often convenient to consider the surface being the surface of a sphere of a radius, r, so that (30) takes the form,

$$P = \lim_{r \to \infty} r^2 \oint d\Omega \boldsymbol{n} \cdot \overline{\boldsymbol{S}} \,, \tag{31}$$

where $d\Omega$ os the elementary solid angle on a unit sphere. Continuing in (SI), according to (22) the power radiated per unit solid angle,

$$\frac{dP}{d\Omega} = \lim_{r \to \infty} r^2 \overline{\boldsymbol{n} \cdot \boldsymbol{E} \times \boldsymbol{H}} \,. \tag{32}$$

One may think of a kind of "radiation friction": you try to accelerate charges yet in response they create fields (radiation) and then this field acts on charges back to oppose your initial intent to accelerate them. Note that in reality the "electricity company" expends most of its power on other kind of friction, i.e. Ohmic losses.

II. GREEN FUNCTIONS FOR THE WAVE EQUATION

Each component satisfies the scalar inhomogeneous wave equation,

$$\nabla^2 \Psi(\boldsymbol{x}, t) - \frac{1}{c^2} \partial_t^2 \Psi(\boldsymbol{x}, t) = -4\pi f(\boldsymbol{x}, t) \tag{33}$$

where $f(\boldsymbol{x}, t)$ is the source distribution. By definition the Green function in a free space satisfies, $G(\boldsymbol{x} - \boldsymbol{x}', t)$ satisfies

$$\nabla^2 G(\boldsymbol{x} - \boldsymbol{x}', t - t') - \frac{1}{c^2} \partial_t^2 G(\boldsymbol{x} - \boldsymbol{x}', t - t') = -4\pi \delta(\boldsymbol{x} - \boldsymbol{x}') \delta(t - t') \tag{34}$$

and defines the wave excited by the sudden and localized perturbation. Consider the Fourier Transformation,

$$G(\boldsymbol{x} - \boldsymbol{x}', t - t') = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega e^{-i\omega t} G_\omega(\boldsymbol{x} - \boldsymbol{x}') \,, \quad G_\omega(\boldsymbol{x} - \boldsymbol{x}') = \int_{-\infty}^{\infty} dt e^{i\omega(t-t')} G(\boldsymbol{x} - \boldsymbol{x}', t - t') \,. \tag{35}$$

Multiply (34) by $e^{i\omega(t-t')}$ and integrate over all times t (by parts in the second term on the right). Then using the definitions (35) one obtains,

$$\nabla^2 G_k(\boldsymbol{x}-\boldsymbol{x}') + k^2 G_k(\boldsymbol{x}-\boldsymbol{x}') = -4\pi\delta(\boldsymbol{x}-\boldsymbol{x}'), \tag{36}$$

where $k \equiv \omega/c$, $G_\omega(\boldsymbol{x}-\boldsymbol{x}') \equiv G_k(\boldsymbol{x}-\boldsymbol{x}')$ and $\omega > 0$. The solution to (36) must be isotropic to respect the rotational symmetry of the space, $G(\boldsymbol{x}-\boldsymbol{x}',t) = G(R,t)$, where $\boldsymbol{R} = \boldsymbol{x}-\boldsymbol{x}'$ and $R = |\boldsymbol{x}-\boldsymbol{x}'|$. And as a result, writing the Laplacian in spherical coordinates transforms (36) into

$$\frac{1}{R}\frac{d^2}{dR^2}[RG_k(R)] + k^2 G_k(R) = -4\pi\delta(\boldsymbol{R}) \tag{37}$$

Lets write (37) in the dimensionless form, with the dimensionless variable $\boldsymbol{\xi} = k\boldsymbol{R}$, $\xi = kR$, and the function $f_k(\xi) \equiv k^{-1}G_k(\xi/k)$. As we will see shortly the function, $f_k(\xi)$ is in fact independent of k so that we write, $f(\xi)$ instead of $f_k(\xi)$ below. Divide (37) by k^3 and notice that,

$$\frac{d}{dR} = \left(\frac{d\xi}{dR}\right)\frac{d}{d\xi} = k\frac{d}{d\xi}, \qquad k^{-3}\delta(\boldsymbol{R}) = \delta(k\boldsymbol{R}) = \delta(\boldsymbol{\xi}). \tag{38}$$

With (38), the equation (37) divided by k^3 takes the form,

$$\frac{1}{\xi}\frac{d^2}{d\xi^2}[\xi f(\xi)] + f(\xi) = -4\pi\delta(\boldsymbol{\xi}) \tag{39}$$

As (39) does not contain k, $f_k(\xi) \equiv f(\xi)$. Consider (39) away from the origin, $\xi \neq 0$,

$$\frac{1}{\xi}\frac{d^2}{d\xi^2}[\xi f(\xi)] + f(\xi) = 0, \quad \xi \neq 0. \tag{40}$$

The function $h(\xi) \equiv \xi f(\xi)$ satisfies the simple ordinary homogeneous equation,

$$\frac{d^2}{d\xi^2}[h(\xi)] + h(\xi) = 0, \quad \xi \neq 0. \tag{41}$$

The general solution of (41) is trivial,

$$h(\xi) = Ae^{+i\xi} + Be^{-i\xi}, \quad \xi \neq 0, \tag{42}$$

and therefore,

$$f(\xi) = A\frac{e^{+i\xi}}{\xi} + B\frac{e^{-i\xi}}{\xi}, \quad \xi \neq 0, \tag{43}$$

where the constants A and B are not yet specified. These can be determined by direct substitution of (43) into (39), but now in the whole space (including the origin, $\xi = 0$). Start with

$$\nabla^2[\frac{e^{\pm i\xi}}{\xi}] = \frac{1}{\xi}\nabla^2[e^{\pm i\xi}] + 2[\boldsymbol{\nabla}e^{\pm i\xi}]\cdot[\boldsymbol{\nabla}\frac{1}{\xi}] + e^{\pm i\xi}\nabla^2[\frac{1}{\xi}] \tag{44}$$

then, with the unit vector, $\hat{\xi} = \boldsymbol{\xi}/\xi$,

$$\frac{1}{\xi}\nabla^2[e^{\pm i\xi}] = \frac{1}{\xi^2}\frac{d^2}{d\xi^2}[\xi e^{\pm i\xi}] = 2(\pm i)\frac{e^{\pm i\xi}}{\xi^2} - \frac{e^{\pm i\xi}}{\xi} \tag{45}$$

$$2[\boldsymbol{\nabla}e^{\pm i\xi}]\cdot[\boldsymbol{\nabla}\frac{1}{\xi}] = 2[(\pm i)e^{\pm i\xi}]\hat{\xi}\cdot[-\frac{1}{\xi^2}]\hat{\xi} = -2(\pm i)\frac{e^{\pm i\xi}}{\xi^2} \tag{46}$$

And finally, as was shown in few different ways in the discussion of electrostatics,

$$\nabla^2[\frac{1}{\xi}] = -4\pi\delta(\boldsymbol{\xi}) \tag{47}$$

Substitution of (45), (46) and (47) into (44) gives

$$\nabla^2\left[\frac{e^{\pm i\xi}}{\xi}\right] = -\frac{e^{\pm i\xi}}{\xi} - 4\pi e^{\pm i\xi}\delta(\boldsymbol{\xi}) = -\frac{e^{\pm i\xi}}{\xi} - 4\pi\delta(\boldsymbol{\xi}) \tag{48}$$

Therefore for $f(\xi)$ given by (43) to be a solution it has to satisfy the restriction,

$$A + B = 1 \tag{49}$$

The choice $A = 1$, $B = 0$ ($A = 0$, $B = 1$) defines the retarded (advanced) Green function for the wave-equation,

$$G_\omega^{\pm}(R) = k\frac{e^{\pm ikR}}{kR} = \frac{e^{\pm ikR}}{R} \tag{50}$$

By taking the inverse Fourier transform, $\tau = t - t'$,

$$G^{(\pm)}(R, \tau) = \int_{-\infty}^{\infty}\frac{d\omega}{2\pi}e^{-i\omega\tau}\frac{e^{\pm i\omega(R/c)}}{R} = \frac{\delta(\tau \mp R/c)}{R} \tag{51}$$

In summary, the two Green functions we have introduced are

$$G^{(\pm)}(\boldsymbol{x} - \boldsymbol{x}', t - t') = \frac{\delta(t - t' \mp c^{-1}|\boldsymbol{x} - \boldsymbol{x}'|)}{|\boldsymbol{x} - \boldsymbol{x}'|} = \frac{\delta(t' - t \pm c^{-1}|\boldsymbol{x} - \boldsymbol{x}'|)}{|\boldsymbol{x} - \boldsymbol{x}'|} = \frac{\delta(t' - t_{r/a})}{|\boldsymbol{x} - \boldsymbol{x}'|}, \tag{52}$$

where the retarded (advanced) time $t_{r/a} = t \mp |\boldsymbol{x} - \boldsymbol{x}'|/c$. This describes the initial disturbance propagating at speed of light as a spherical shock-wave. Note that

$$G^{(+)}(\boldsymbol{x} - \boldsymbol{x}', t < t') = 0, \quad G^{(-)}(\boldsymbol{x} - \boldsymbol{x}', t > t') = 0, \tag{53}$$

and as a result $G^{(+)}$ describes the light cone pointing down (if the time axis is going up) that appears first at $t = t' + 0^+$ and $\boldsymbol{x} = \boldsymbol{x}'$ and opens up towards the future. This retarded light cone exists at times $t > t'$. Such a light cone is Lorentz invariant. To the contrary, $G^{(-)}$ describes the (advanced) light cone pointing down which exists for times t that are in the past relative to t'. Both cones have a common vertex at the flash event located at ($\boldsymbol{x} = \boldsymbol{x}'$, $t = t'$) in space-time. They otherwise are disjoint.

The two solution are time reversal partners, (52)

$$G^{(-)}(\boldsymbol{x} - \boldsymbol{x}', t - t') = G^{(+)}(\boldsymbol{x} - \boldsymbol{x}', -(t - t')) \tag{54}$$

The presence of the two partner solutions (54) follows as the original equation, (34) is of the *second* order in time derivatives and has a time reversal invariant source (indeed $\delta(t - t') = \delta(t' - t)$). [Note parenthetically that the Schrödinger equation is time reversal invariant as well even though its first order in time derivatives. But that is because the wave function in quantum mechanics is complex and the first derivative comes with the complex i].

A. Lorentz invariance

The separation into the two kinds of the Green functions is Lorentz invariant concept. This is nothing but the statement of the concepts of past and future being absolute for two causally related events that we have discussed in the framework of relativity. Lets assign the superscript $K(\bar{K})$ to the quantities in the two reference frames. Imagine, $t^K - t'^K - c^{-1}|\boldsymbol{x}^K - \boldsymbol{x}'^K| = 0$ so that $t^K - t'^K > 0$, i.e. the event $(ct^K, \boldsymbol{x}^K)$ lies on the *upper* light cone originating at the event, $(ct'^K, \boldsymbol{x}'^K)$ in K. In any other frame $\bar{K}$ we are assured that $(t^{\bar{K}} - t'^{\bar{K}})^2 - c^{-2}|\boldsymbol{x}^{\bar{K}} - \boldsymbol{x}'^{\bar{K}}|^2 = 0$ as the interval between the events is Lorentz invariant. Let us also show that the sign of $t^{\bar{K}} - t'^{\bar{K}}$ is the same as the sign of $t^K - t'^K$. Lets say the two frames share a common coordinate axes, and $\bar{K}$ moves along $\hat{x}$ with a velocity, V. Then denoting $\tanh\eta = V/c$, $-\infty < \eta < \infty$ the Lorentz transformation reads,

$$\begin{aligned}
ct^{\bar{K}} &= ct^K \cosh\eta - (x_1)^K \sinh\eta \\
(x_1)^{\bar{K}} &= -ct^K \sinh\eta + (x_1)^K \cosh\eta \\
(x_{2,3})^{\bar{K}} &= (x_{2,3})^K
\end{aligned} \tag{55}$$

with the same relationship between the primed quantities. We then have

$$ct^{\bar{K}} - t'^{\bar{K}} = c(t^K - t'^K)\cosh\eta - [(x)_1^K - (x')_1^K]\sinh\eta \geq c(t^K - t'^K)\cosh\eta - |[(x)_1^K - (x')_1^K]|\sinh\eta$$
$$\geq c(t^K - t'^K)\cosh\eta - |\boldsymbol{x}^K - \boldsymbol{x}'^K|\sinh\eta = c(t^K - t'^K)[\cosh\eta - \sinh\eta] = c(t^K - t'^K)e^{-\eta} > 0 \,. \qquad (56)$$

So we have proved that on a light cone, the relative past and future are Lorentz invariant concepts. This observation allows us to prove the Lorentz invariance of the Green function itself. Here to simplify the discussion, we focus on the **scalar** wave equation (34). In the EM it is slightly more complicated as the Green function describes the response of one 4-vector (4-potential, A^μ) to the the other 4-vector (4-current, J^μ) and as such is (*gauge dependent!*) 4-tensor, $G^{\mu\nu}(\boldsymbol{x} - \boldsymbol{x}', t - t')$ that happens to be diagonal in the Lorentz gauge.

Lets get back to the case of the scalar wave equation (33) with f being a scalar function. Consider the two kinds of Green functions, (52) written in equivalent way as follows,

$$G^{(\pm)}(\boldsymbol{x} - \boldsymbol{x}', t - t') = \frac{\delta(t - t' \mp c^{-1}|\boldsymbol{x} - \boldsymbol{x}'|)}{|\boldsymbol{x} - \boldsymbol{x}'|} = \frac{\theta^\pm(t - t')}{|\boldsymbol{x} - \boldsymbol{x}'|}\delta\left[\frac{(t - t' \mp c^{-1}|\boldsymbol{x} - \boldsymbol{x}'|)(t - t' \pm c^{-1}|\boldsymbol{x} - \boldsymbol{x}'|)}{(t - t' \pm c^{-1}|\boldsymbol{x} - \boldsymbol{x}'|)}\right]$$

$$= \frac{\theta^\pm(t - t')}{|\boldsymbol{x} - \boldsymbol{x}'|}\delta\left[\frac{(t - t' \mp c^{-1}|\boldsymbol{x} - \boldsymbol{x}'|)(t - t' \pm c^{-1}|\boldsymbol{x} - \boldsymbol{x}'|)}{(\pm 2c^{-1}|\boldsymbol{x} - \boldsymbol{x}'|)}\right]$$

$$= \frac{\theta^\pm(t - t')}{|\boldsymbol{x} - \boldsymbol{x}'|}\delta\left[\frac{(t - t')^2 - c^{-2}(\boldsymbol{x} - \boldsymbol{x}')^2}{(\pm 2c^{-1}|\boldsymbol{x} - \boldsymbol{x}'|)}\right]$$

$$= \frac{\theta^\pm(t - t')|\pm 2c^{-1}|\boldsymbol{x} - \boldsymbol{x}'||}{|\boldsymbol{x} - \boldsymbol{x}'|}\delta\left[(t - t')^2 - c^{-2}(\boldsymbol{x} - \boldsymbol{x}')^2\right]$$

$$= 2c\theta^\pm(t - t')\delta\left[c^2(t - t')^2 - (\boldsymbol{x} - \boldsymbol{x}')^2\right] \,, \qquad (57)$$

where $\theta^\pm(x) \equiv \theta(\pm x)$, and $\theta(x)$ is the usual step-function and we used the property of the delta function, $|a|\delta(ax) = \delta(x)$. It follows from (57) that the Green functions are product of the two Lorentz invariant factors: 1) the step function, $\theta^\pm(t - t')$ is Lorentz invariant because the order of times t and t' cannot be flipped thanks to the Eq. (56). 2) The second factor is the delta-function of the **interval**, and is Lorentz invariant by definition.

1. An example

To make the discussion less abstract consider the special case of the two events on a single light-cone, $(\boldsymbol{x}, t)$ and $(\boldsymbol{x}', t')$ as before, but with $(x_{2,3})^K - (x'_{2,3})^K = (x_{2,3})^{\bar{K}} - (x'_{2,3})^{\bar{K}} = 0$. In other words we consider both the initial flash and the subsequent observation of the signal to occur on the x-axis in both reference frames. Take for definiteness, $t_1 > t'_1$ and $x_1 > x'_1$ in $\bar{K}$. For this particular case, denoting $c^{-1}[(x_1)^K - (x'_1)^K] = \Delta x^K$ and $(t)^K - (t')^K = \Delta t^K$ with the same definitions in the $\bar{K}$ frame. Then (52) gives

$$[G^{(\pm)}]^{\bar{K}}(\boldsymbol{x}_{\bar{K}} - \boldsymbol{x}'_{\bar{K}}, t_{\bar{K}} - t'_{\bar{K}}) = c^{-1}\frac{\delta[\Delta t^{\bar{K}} - \Delta x^{\bar{K}}]}{\Delta x^{\bar{K}}} = c^{-1}\frac{\delta[\Delta t^K\cosh\eta - \Delta x^K\sinh\eta - (\Delta x^K\cosh\eta - \Delta t^K\sinh\eta)]}{\Delta x^K\cosh\eta - \Delta t^K\sinh\eta}$$

$$= c^{-1}\frac{\delta[(\Delta x^K - \Delta t^K)(\cosh\eta + \sinh\eta)]}{\Delta x^K\cosh\eta - \Delta t^K\sinh\eta} = c^{-1}\frac{\delta[(\Delta x^K - \Delta t^K)(\cosh\eta + \sinh\eta)]}{\Delta x^K(\cosh\eta - \sinh\eta)}$$

$$= c^{-1}\frac{\delta[(\Delta x^K - \Delta t^K)]}{\Delta x^K(\cosh\eta - \sinh\eta)(\cosh\eta + \sinh\eta)} = c^{-1}\frac{\delta[(\Delta x^K - \Delta t^K)]}{\Delta x^K(\cosh^2\eta - \sinh^2\eta)} = c^{-1}\frac{\delta[(\Delta x^K - \Delta t^K)]}{\Delta x^K}$$

$$= [G^{(\pm)}]^K(\boldsymbol{x}_K - \boldsymbol{x}'_K, t_K - t'_K) \qquad (58)$$

The last equality is expected: the Dalambertian, $\Box \equiv c^{-2}\partial_t^2 - \nabla^2$ is Lorentz invariant, as well as the delta function. Therefore all observer must solve exactly the same equation, (34) but each one using its own coordinate notations. As a result all observers will come up with the same solution, as is explicitly demonstrated by (58).

B.   Boundary Values Problem

We are going to complement the wave equation (33) by the physically motivated boundary conditions. The physical situation of interest is the case when the sources are non-existent at remote past,

$$f(\boldsymbol{x}, t = -\infty) = 0. \tag{59}$$

In this case we require that at $t = -\infty$ the solution is

$$\Psi(\boldsymbol{x}, t \to -\infty) = \Psi_{in}(\boldsymbol{x}, t), \tag{60}$$

where $\Psi_{in}(\boldsymbol{x}, t)$ is a given solution of the *homogeneous* wave equation. In this situation the solution satisfying the boundary conditions and the differential equation reads,

$$\Psi(\boldsymbol{x}, t) = \Psi_{in}(\boldsymbol{x}, t) + \int d^3x' \int_{-\infty}^{\infty} dt' G^{(+)}(\boldsymbol{x}, t; \boldsymbol{x}', t') f(\boldsymbol{x}', t') \tag{61}$$

The boundary condition, (59) is satisfied by (62) in virtue of the properties, (53). Normally, we would be interested in the special case, $\Psi_{in}(\boldsymbol{x}, t) = 0$. In this case using the result (52) and performing the t' integration gives,

$$\Psi(\boldsymbol{x}, t) = \int d^3x' \frac{f(\boldsymbol{x}', t_r)}{|\boldsymbol{x} - \boldsymbol{x}'|} = \int d^3x' \frac{f(\boldsymbol{x}', t - c^{-1}|\boldsymbol{x} - \boldsymbol{x}'|)}{|\boldsymbol{x} - \boldsymbol{x}'|} \tag{62}$$

1.   Source that is harmonic in time

Here we consider the sources that are harmonic functions of time, and are hence characterized by a fixed frequency ω and the corresponding wave-number, $k = \omega/c$. In other words now ω is a fixed parameter. This is particularly important type of source of the radiation,

$$f(\boldsymbol{x}, t) = f(\boldsymbol{x}) \cos(\omega t - \theta) = \mathrm{Re}[f_\omega(\boldsymbol{x}) e^{-i\omega t}], \, f_\omega(\boldsymbol{x}) = f(\boldsymbol{x}) e^{i\theta}, \tag{63}$$

where $f(\boldsymbol{x})$ is a real function, θ is an arbitrary phase, and $f_\omega(\boldsymbol{x})$ is a complex source amplitude. Substitute (63) in (62),

$$\Psi(\boldsymbol{x}, t) = \int d^3x' \frac{\mathrm{Re}\{f_\omega(\boldsymbol{x}') e^{-i\omega t + ik|\boldsymbol{x} - \boldsymbol{x}'|}\}}{|\boldsymbol{x} - \boldsymbol{x}'|} \tag{64}$$

Define the complex amplitude of the solution,

$$\Psi(\boldsymbol{x}, t) = \mathrm{Re}\{\Psi_\omega(\boldsymbol{x}) e^{-i\omega t}\} \tag{65}$$

Comparison of (64) and (65) gives

$$\Psi_\omega(\boldsymbol{x}) = \int d^3x' f_\omega(\boldsymbol{x}') \frac{e^{ik|\boldsymbol{x} - \boldsymbol{x}'|}}{|\boldsymbol{x} - \boldsymbol{x}'|} \tag{66}$$

2.   Asymptotic scaling of solutions far from the sources

The choice of the **retarded** rather than the *advanced* Green functions makes the asymptotic (large distance) dependence of solution (64) on space and time to be **OUTGOING** rather than *INCOMING* wave. In other words, the scaling for the retarded solutions is $\cos(\omega t - kx)$ and for the advanced solutions, the scaling is different $\cos(\omega t + kx)$.

This difference is of fundamental importance. It stems from the underlying difference in the time dependence of $G^{(+)}$ and $G^{(-)}$, (51). The retarded Green function, $G^{(+)}$ kicks off at $t = t'$ after which it describes propagation of energy **outwards** away from the disturbance, localised at $(\boldsymbol{x}', t')$. In contrast, the advanced Green function, $G^{(-)}$ describes the shrinkage of the wave front from being infinitely large in remote past to a single point of a space-time, $(\boldsymbol{x}', t')$ where the localised disturbance flashes. As a result $G^{(-)}$ describes the energy flux carried with the incoming wave front, **inwards**.

The above distinction allows us to treat the stationary situations by choosing among the infinite set of possible solutions those that are outgoing waves (or their superpositions) at large distances.

C. Solution of the wave equation, (33) of electrodynamics

For the problems with the sources gradually turned on from (or absent at) $t' = -\infty$ and with the solution specified at $t' = -\infty$ (in this case the latter is a solution of the homogeneous wave equation) the solution at any finite time t reads,

$$\Phi(\boldsymbol{x}, t) = \frac{1}{4\pi\epsilon_0} \int \frac{d^3 x'}{|\boldsymbol{x} - \boldsymbol{x}'|} \rho(\boldsymbol{x}', t - c^{-1}|\boldsymbol{x} - \boldsymbol{x}'|), \qquad \boldsymbol{A}(\boldsymbol{x}, t) = \frac{\mu_0}{4\pi} \int \frac{d^3 x'}{|\boldsymbol{x} - \boldsymbol{x}'|} \boldsymbol{J}(\boldsymbol{x}', t - c^{-1}|\boldsymbol{x} - \boldsymbol{x}'|) \qquad (67)$$

Alternatively the choice of the retarded Green function amounts to having the **outgoing** solutions. Indeed, (67) becomes a function of the combination $t - |\boldsymbol{x}|/c$, in other words describe the **outgoing** wave front.

1. Another example: Field of a moving charge revisited

In this example we compute the potentials due to the moving charge from (67). Lets imagine a point charge, q moves along the x-axis with velocity v in some given reference frame. Assume for definiteness that it passes the origin at time $t = 0$. Then the source charge and current densities read,

$$\rho(\boldsymbol{x}, t) = q\delta(x - vt)\delta(y)\delta(z), \quad \boldsymbol{J}(\boldsymbol{x}, t) = qv\hat{x}\delta(x - vt)\delta(y)\delta(z) \qquad (68)$$

It is sufficient to compute the potential Φ as the vector potential requires exactly the same calculation. Denoting $\rho = \sqrt{y^2 + z^2}$, $\beta = v/c$, $\bar{x} = x - vt$, we write (67),

$$\Phi(x, y, z, t) = \frac{q}{4\pi\epsilon_0} \int \frac{dx' dy' dz'}{\sqrt{(y - y')^2 + (z - z')^2 + (x - x')^2}} \delta\left[x' - vt - \beta\sqrt{(x - x')^2 + (y - y')^2 + (z - z')^2}\right] \delta(y')\delta(z')$$

$$= \frac{q}{4\pi\epsilon_0} \int \frac{dx'}{\sqrt{\rho^2 + (x - x')^2}} \delta\left[x' - vt - \beta\sqrt{(x - x')^2 + \rho^2}\right]$$

$$= \frac{q}{4\pi\epsilon_0} \int \frac{dx'}{\sqrt{\rho^2 + (\bar{x} + vt - x')^2}} \delta\left[x' - vt - \beta\sqrt{(\bar{x} + vt - x')^2 + \rho^2}\right] \qquad (69)$$

Introduce a new integration variable, $y = x' - \bar{x} - vt$ to write (69) as

$$\Phi(\bar{x} = x - vt, \rho) = \frac{q}{4\pi\epsilon_0} \int \frac{dy}{\sqrt{\rho^2 + y^2}} \delta\left[y + \bar{x} - \beta\sqrt{y^2 + \rho^2}\right] \qquad (70)$$

To evaluate the integral (70) introduce y_* satisfying,

$$y_* + \bar{x} - \beta\sqrt{y_*^2 + \rho^2} = 0 \qquad (71)$$

Then, according to the properties of the δ function,

$$\Phi(\bar{x} = x - vt, \rho) = \frac{q}{4\pi\epsilon_0} \frac{1}{\sqrt{\rho^2 + y_*^2}} \left| \frac{d}{dy_*}(y_* + \bar{x} - \beta\sqrt{y_*^2 + \rho^2}) \right|^{-1} = \frac{q}{4\pi\epsilon_0} \frac{1}{\sqrt{\rho^2 + y_*^2}} \overbrace{\frac{1}{1 - \beta\frac{y_*}{\sqrt{\rho^2 + y_*^2}}}}^{>0!}$$

$$= \frac{q}{4\pi\epsilon_0} \frac{1}{\sqrt{\rho^2 + y_*^2} - \beta y_*} = \frac{q}{4\pi\epsilon_0} \frac{\beta}{\beta\sqrt{\rho^2 + y_*^2} - \beta^2 y_*} \overset{(71)}{=} \frac{\beta}{y_* + \bar{x} - \beta^2 y_*} = \frac{\beta}{\bar{x} + y_*(1 - \beta^2)} \qquad (72)$$

Lets solve (71),

$$y_*^2 + 2y_*\bar{x} + \bar{x}^2 = \beta^2(y_*^2 + \rho^2), \quad y_*^2(1 - \beta^2) + 2y_*\bar{x} + \bar{x}^2 - \beta^2\rho^2 = 0 \qquad (73)$$

which gives two roots

$$y_* = \frac{1}{1-\beta^2}\left[-\bar{x} \pm \sqrt{\bar{x}^2 - (1-\beta^2)(\bar{x}^2 - \beta^2\rho^2)}\right] \tag{74}$$

yet only the upper sign $+$ is consistent with (71) which then leads to

$$y_*(1-\beta^2) + \bar{x} = \sqrt{\bar{x}^2 - (1-\beta^2)(\bar{x}^2 - \beta^2\rho^2)} = \beta\sqrt{\bar{x}^2 + (1-\beta^2)\rho^2} \tag{75}$$

Now substitute (75) in (72)

$$\Phi(\bar{x} = x - vt, \rho) = \frac{q}{4\pi\epsilon_0}\frac{1}{\sqrt{\bar{x}^2 + (1-\beta^2)\rho^2}} = \frac{q}{4\pi\epsilon_0}\frac{1}{\sqrt{(x-vt)^2 + (1-\beta^2)\rho^2}}$$

$$= \frac{q}{4\pi\epsilon_0}\frac{1}{\sqrt{1-\beta^2}}\frac{1}{\sqrt{(x-vt)^2(1-\beta^2)^{-1} + \rho^2}} = \boxed{\frac{q\gamma}{4\pi\epsilon_0}\frac{1}{\sqrt{\gamma^2(x-vt)^2 + \rho^2}}} \tag{76}$$

where the standard notation $\gamma = (1-\beta^2)^{-1/2}$ has been used. (76) gives the right answer.

III.   DIPOLE APPROXIMATION

We are going to analyze Eq. (67) under different sets of approximations.
Approximations, d is the typical system dimension:

1. $d/r \ll 1$, $\boldsymbol{x} = r\boldsymbol{n}$, where $\boldsymbol{n} = (n_1, n_2, n_3) = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$ is the direction of observation form the radiating system,

$$|\boldsymbol{x} - \boldsymbol{x}'| \approx r - \boldsymbol{n}\boldsymbol{x}' \tag{77}$$

$$\frac{1}{|\boldsymbol{x} - \boldsymbol{x}'|} \approx \frac{1}{r} + \frac{\boldsymbol{n}\boldsymbol{x}'}{r^2} \tag{78}$$

Then (67) can be approximated to the leading (zeroth) order in d/r, as

$$\boldsymbol{A}(\boldsymbol{x}, t) \approx \frac{\mu_0}{4\pi r}\int d^3x'\, \boldsymbol{J}(\boldsymbol{x}', t - c^{-1}|\boldsymbol{x} - \boldsymbol{x}'|) \tag{79}$$

2. $d/\lambda \ll 1$. Let the typical time scale of variation of $\boldsymbol{J}(\boldsymbol{x}', t)$ in (67) is T. For the periodic motion of frequency ω, $T \sim 1/\omega$. If $\boldsymbol{n}\boldsymbol{x}'/c \sim d/c \ll T$ is shorter than this time we have,

$$\boldsymbol{J}(x', t - c^{-1}|x - x'|) \approx \boldsymbol{J}(x', t - r/c) \equiv \boldsymbol{J}(x', t') \tag{80}$$

to the leading order in $d/Tc \sim \omega d/c = kd \ll 1$. Alternatively, $d/T \sim v$, so the condition is $v/c \ll 1$, i.e. the motion of particles in the radiating system is non-relativistic.

Dipole approximation is obtained to the leading (zeroth) order in both d/r and kd of (67),

$$\boldsymbol{A}(x, t) = \frac{\mu_0}{4\pi r}\int d^3x'\, \boldsymbol{J}(x', t'), \quad t' = t - r/c \tag{81}$$

Using (13), (81) becomes

$$\boldsymbol{A}(\boldsymbol{x}, t) = \frac{\mu_0}{4\pi r}\dot{\boldsymbol{p}}(t - r/c) \tag{82}$$

A.   Fields in radiation zone, $kr \gg 1$

Lets find fields in the radiation zone, $kr \gg 1$. Start with the magnetic field, following from the approximation (82),

$$\boldsymbol{H} = \mu_0^{-1} \nabla \times \boldsymbol{A} = \nabla \times [\frac{1}{4\pi r} \dot{\boldsymbol{p}}(t - r/c)]. \tag{83}$$

When derivatives in (83) act on the $1/r$ prefactor in (82) terms of the order $\sim d/r^2$ are generated, while when derivatives in a curl operator acts on a $\dot{d}$ prefactor terms of the order $\sim kd/r$ are obtained. In radiation zone only these latter terms are important, and as a result (83) simplifies,

$$\boldsymbol{H} \approx \frac{1}{4\pi r} \nabla \times \dot{\boldsymbol{p}}(t - r/c). \tag{84}$$

Use

$$\nabla \times \boldsymbol{F}(|\boldsymbol{x}|) = \boldsymbol{n} \times \frac{d\boldsymbol{F}}{d|\boldsymbol{x}|} \tag{85}$$

to write ($\boldsymbol{x} \equiv \boldsymbol{n} r$)

$$\nabla \times \dot{\boldsymbol{p}}(t - r/c) = \boldsymbol{n} \times \frac{d}{dr} \dot{\boldsymbol{p}} = c^{-1} \ddot{\boldsymbol{p}} \times \boldsymbol{n} \tag{86}$$

Substituting (86) in (84) one gets,

$$\boldsymbol{H}(\boldsymbol{x}, t) = \frac{1}{4\pi r c} \ddot{\boldsymbol{p}}(t - r/c) \times \boldsymbol{n}, \qquad \boldsymbol{x} \equiv \boldsymbol{n} r \tag{87}$$

The dipole moment in (87) is evaluated at the time $t' = t - r/c$. To find electric field without resorting to a scalar potential, we can use

$$\frac{\partial \boldsymbol{D}}{\partial t} = \nabla \times \boldsymbol{H} \tag{88}$$

in the absence of free charges, which is certainly true at $r \gg d$. This way we cannot determine the static part of $\boldsymbol{E}$. But this part is not related to radiation. In the radiation zone we find by the same arguments leading to (87) gives using (85) again,

$$4\pi c \frac{\partial \boldsymbol{D}}{\partial t} = \nabla \times [\ddot{\boldsymbol{p}}(t - r/c) \times \frac{\boldsymbol{n}}{r}] = \ddot{\boldsymbol{p}}(t - r/c)(\nabla \cdot \frac{\boldsymbol{n}}{r}) - \frac{\boldsymbol{n}}{r}(\nabla \cdot \ddot{\boldsymbol{p}}(t - r/c)) + (\frac{\boldsymbol{n}}{r} \cdot \nabla)\ddot{\boldsymbol{p}}(t - r/c) - (\ddot{\boldsymbol{p}}(t - r/c) \cdot \nabla)\frac{\boldsymbol{n}}{r}$$
$$\approx -\frac{\boldsymbol{n}}{r}(\nabla \cdot \ddot{\boldsymbol{p}}(t - r/c)) + (\frac{\boldsymbol{n}}{r} \cdot \nabla)\ddot{\boldsymbol{p}}(t - r/c) \tag{89}$$

Further using the relationships, recall again $\boldsymbol{x} = r\hat{n}$,

$$\frac{\partial |\boldsymbol{x}|}{\partial x_l} = [\boldsymbol{n}]_l, \qquad \nabla_l \ddot{p}_k(t - r/c) = \frac{\partial \ddot{p}_k(t - r/c)}{\partial r} \frac{\partial r}{\partial x_l} = -\frac{n_l}{c} \dddot{p}_k(t - r/c) \tag{90}$$

we obtain,

$$\frac{\partial \boldsymbol{D}}{\partial t} = \frac{1}{4\pi r c^2} \boldsymbol{n} \times (\boldsymbol{n} \times \dddot{\boldsymbol{p}}) \tag{91}$$

Integrating (91) and discarding constant in time component of the displacement write,

$$\boldsymbol{E} = \epsilon_0^{-1} \boldsymbol{D} = \frac{1}{4\pi \epsilon_0 r c^2} \boldsymbol{n} \times (\boldsymbol{n} \times \ddot{\boldsymbol{p}}) = Z_0 \boldsymbol{H} \times \boldsymbol{n}, \, Z_0 = \sqrt{\mu_0 / \epsilon_0}$$
$$\boldsymbol{E} = c\boldsymbol{B} \times \boldsymbol{n} \tag{92}$$

The dipole moment is evaluated at the time $t' = t - r/c$. (92) and (87) specify fields in the radiation zone in the dipole approximation. In both cases the second derivatives enters so that only accelerating charges radiate, as also is clear from the relativity principle, since resting charge is not radiating for sure. To recover the static part of electric field we write to the leading order in $v/c \sim kd$, for the Φ part of (67),

$$\Phi(\boldsymbol{x}, t) = \frac{q(t' = t - r/c)}{4\pi \epsilon_0 r} \tag{93}$$

is indeed time independent as the total charge $q(t) = q$ is constant. The expansion in kd would give terms of higher order beyond the leading one. The expansion of potentials is alternative way to obtain the dipole approximation.

B. Radiation Intensity

According to (32), and with (92),

$$\frac{dP}{d\Omega} = r^2 Z_0 \overline{\boldsymbol{n} \cdot [(\boldsymbol{H} \times \boldsymbol{n}) \times \boldsymbol{H}]} \tag{94}$$

As $(\boldsymbol{H} \times \boldsymbol{n}) \times \boldsymbol{H} = \boldsymbol{n}H^2 - \boldsymbol{H}(\boldsymbol{n} \cdot \boldsymbol{H})$, we get $\boldsymbol{n} \cdot [(\boldsymbol{H} \times \boldsymbol{n}) \times \boldsymbol{H}] = H^2 - (\boldsymbol{n} \cdot \boldsymbol{H})^2 = H^2 \sin^2 \theta$, where θ is the angle between $\boldsymbol{H}$ and $\boldsymbol{n}$, but in the dipole approximation (in fact generally in the radiation zone) $|\theta| = \pi/2$, see Eq. (87), and

$$\frac{dP}{d\Omega} = r^2 Z_0 \overline{H^2} \tag{95}$$

C. Some examples

1. Oscillating dipole

Consider an oscillating dipole moment (charged oscillator),

$$\boldsymbol{p}(t) = \boldsymbol{p}_\omega \cos(\omega t) \tag{96}$$

with real constant vector $\boldsymbol{p}_\omega = p\hat{z}$. The magnetic field, (87),

$$\boldsymbol{H}(t) = -\frac{\omega^2}{4\pi rc}\boldsymbol{p}(t - r/c) \times \boldsymbol{n} = \frac{\omega^2}{4\pi rc}\boldsymbol{n} \times \boldsymbol{p}_\omega \cos(\omega t - kr) \tag{97}$$

since from (96), $\boldsymbol{p}(t) \times \boldsymbol{n} = \cos(\omega t)\boldsymbol{p}_\omega \times \boldsymbol{n}$. The magnetic field, (97) in polar coordinates, from (A6),

$$\boldsymbol{H} = -\hat{\phi}\sin\theta\frac{\omega^2}{4\pi rc}p\cos(\omega t - kr) \tag{98}$$

and the polarization, (92),

$$\boldsymbol{E} = -\hat{\theta}\sin\theta\frac{Z_0\omega^2}{4\pi rc}p\cos(\omega t - kr) \tag{99}$$

(98),(99) represent a linear polarized wave.

By (95) the power

$$\frac{dP}{d\Omega} = \frac{r^2 Z_0 \omega^4}{(4\pi rc)^2}(\boldsymbol{p} \times \boldsymbol{n})^2\overline{\cos^2(\omega t)} \tag{100}$$

With $\omega = ck$,

$$\frac{dP}{d\Omega} = \frac{Z_0 c^2 k^4}{32\pi^2}p^2\sin^2(\theta_{\boldsymbol{p},\boldsymbol{n}})\,. \tag{101}$$

The total intensity, $< \sin^2 \theta > = 1- < \cos^2 \theta > = 1 - 1/3 = 2/3$,

$$P = \frac{Z_0 c^2 k^4}{12\pi}p^2 = \frac{Z_0\epsilon_0 c^2 k^4}{12\pi\epsilon_0}p^2 = \frac{ck^4}{12\pi\epsilon_0}p^2 = \frac{\omega^4}{12\pi\epsilon_0 c^3}p^2\,. \tag{102}$$

2. Rotating dipole

As a next example we consider the rotating dipole. Assume the rotation plane is xy,

$$\boldsymbol{p} = p\left[\hat{x}\cos(\omega t) + \hat{y}\sin(\omega t)\right] \tag{103}$$

Using (A7) in (87) we get for the magnetic field,

$$\boldsymbol{H} = -\frac{\omega^2 p}{4\pi rc}[\hat{\phi}\cos\theta\cos(\phi - \omega t_r) + \hat{\theta}\sin(\phi - \omega t_r)]\,, t_r = t - r/c \tag{104}$$

Then with (92),

$$\boldsymbol{E} = -Z_0\frac{\omega^2 p}{4\pi rc}[\hat{\theta}\cos\theta\cos(\phi - \omega t_r) - \hat{\phi}\sin(\phi - \omega t_r)] \tag{105}$$

Consider a few directions. For $\boldsymbol{n} = \hat{z}$, $\phi = 0$, $\theta \to 0$, $\hat{\theta} \to \hat{x}$, $\hat{\phi} \to \hat{y}$, and

$$\boldsymbol{H} = \frac{\omega^2 p}{4\pi rc}[\hat{x}\sin(\omega t_r) - \hat{y}\cos(\omega t_r)]\,, \quad \boldsymbol{E} = -Z_0\frac{\omega^2 p}{4\pi rc}[\hat{x}\cos(\omega t_r) + \hat{y}\sin(\omega t_r)] \tag{106}$$

This is circularly polarized wave: electric field vector rotates in the plane perpendicular to the direction of propagation $\boldsymbol{n} = \hat{z}$ and the magnetic field stays perpendicular to it.

Consider the direction $\boldsymbol{n} = \hat{x}$. $\phi = 0$, $\theta = \pi/2$, $\hat{\theta} = -\hat{z}$, $\hat{\phi} = \hat{y}$. And

$$\boldsymbol{H} = -\frac{\omega^2 p}{4\pi rc}\hat{z}\sin(\omega t_r)\,, \quad \boldsymbol{E} = -Z_0\frac{\omega^2 p}{4\pi rc}\hat{y}\sin(\omega t_r) \tag{107}$$

That is the linearly polarized wave with the polarization vector along $\hat{y}$, Note that the plane wave relation holds,

$$\boldsymbol{H} = Z_0^{-1}\boldsymbol{n} \times \boldsymbol{E} \tag{108}$$

In both cases the spherical wave looks locally like a plane wave. This is a general feature of radiation field in radiation zone.

Radiation power from (104), (105) in (32) we get,

$$\frac{dP}{d\Omega} = \left(\frac{\omega^2 p}{4\pi c}\right)^2 Z_0\frac{1 + \cos^2\theta}{2} \tag{109}$$

The total power radiated is then,

$$P = \oint d\Omega\frac{dP}{d\Omega} = \frac{Z_0 c^2 k^4}{6\pi}p^2 \tag{110}$$

is twice as large compared to (102). It is understood as the rotating dipole can be thought of as a superposition of two perpendicular oscillating dipoles of the same amplitude but perpendicular and shifted by $\pi/2$ phase in time. Consider the direction $\boldsymbol{n} = \hat{z}$. Clearly the electric field of one constituent dipoles is parallel to the magnetic one of the other and vice versa. So the intensities should simply add up. Indeed the twice of (101) for $\theta_{n,p} = \pi/2$ is (109) at $\theta = 0$.

For $\boldsymbol{n} = \hat{x}$, (in-plane) on the other hand one (98) and (99) show that one of the oscillators is not producing a field at all. So the power is supplied only by one of the oscillators that's why for $\theta = \pi/2$ in (109) we get the same result as from (101) with $\theta_{n,p} = \pi/2$.

D.   Harmonic fields

In the case of harmonic fields all the quantities vary with time according to $\sim e^{-i\omega t}$ with the real part assumed to be taken at the end of the calculation, so that

$$\rho(\boldsymbol{x}, t) = \rho(\boldsymbol{x})e^{-i\omega t}\,, \quad \boldsymbol{J}(\boldsymbol{x}, t) = \boldsymbol{J}(\boldsymbol{x})e^{-i\omega t} \tag{111}$$

Quite generally, Substituting (111) in (67) and integrating over t' we obtain,

$$\boldsymbol{A}(\boldsymbol{x}, t) = \boldsymbol{A}(\boldsymbol{x})e^{-i\omega t}\,, \tag{112}$$

where

$$\boldsymbol{A}(\boldsymbol{x}) = \frac{\mu_0}{4\pi}\int d^3 x'\frac{\boldsymbol{J}(\boldsymbol{x}')}{|\boldsymbol{x} - \boldsymbol{x}'|}e^{ik|\boldsymbol{x} - \boldsymbol{x}'|}\,, \quad k \equiv \omega/c\,. \tag{113}$$

For the harmonic fields we also have,

$$-i\omega \boldsymbol{D}(\boldsymbol{x}) = \nabla \times \boldsymbol{H}(\boldsymbol{x}) \tag{114}$$

for the (complex) amplitudes.

In the dipole approximation we get

$$\boldsymbol{A}(\boldsymbol{x}) = \frac{\mu_0}{4\pi} \frac{e^{ikr}}{r} \int d^3x' \boldsymbol{J}(\boldsymbol{x}')\,, \qquad (J.D.J.(9.13)) \tag{115}$$

Using the continuity equation,

$$i\omega\rho(x) = \nabla \boldsymbol{J} \tag{116}$$

and using the integration by parts

$$\int d^3x' \boldsymbol{J} = -\int d^3x' \boldsymbol{x}' (\nabla' \cdot \boldsymbol{J}) = -i\omega \int d^3x' \boldsymbol{x}' \rho(\boldsymbol{x}') = -i\omega \boldsymbol{p}_\omega \tag{117}$$

Then the vector potential,

$$\boldsymbol{A}(\boldsymbol{x}) = -\frac{i\mu_0\omega}{4\pi} \boldsymbol{p}_\omega \frac{e^{ikr}}{r}\,, \tag{118}$$

which allows calculation of the magnetic field, to the leading order in both small parameters, d/r and d/λ both in near and far zones as well as in between. Computing the curl of (118) and using the (85)

$$\boldsymbol{H} = \mu_0^{-1} \nabla \times \boldsymbol{A}(\boldsymbol{x}) = \frac{ck^2}{4\pi} (\boldsymbol{n} \times \boldsymbol{p}_\omega) \frac{e^{ikr}}{r} \left(1 - \frac{1}{ikr}\right) \tag{119}$$

Intensity in terms of the harmonic fields, averaged over a period, $T = 2\pi/\omega$,

$$\boldsymbol{S} = \overline{\boldsymbol{E} \times \boldsymbol{H}} = \overline{\mathrm{Re}[\boldsymbol{E}] \times \mathrm{Re}[\boldsymbol{H}]} = \overline{\frac{1}{2}(\boldsymbol{E}_\omega e^{-i\omega t} + \boldsymbol{E}_\omega^* e^{+i\omega t}) \times \frac{1}{2}(\boldsymbol{H}_\omega e^{-i\omega t} + \boldsymbol{H}_\omega^* e^{+i\omega t})}$$

$$= \frac{1}{4}\overline{\boldsymbol{E}_\omega \times \boldsymbol{H}_\omega^* + \boldsymbol{E}_\omega^* \times \boldsymbol{H}_\omega} = \frac{1}{2}\mathrm{Re}\overline{\boldsymbol{E}_\omega \times \boldsymbol{H}_\omega^*}\,. \tag{120}$$

Note the extra factor of 1/2 which has the same origin as the 1/2 from averaging of $\cos^2(t_r)$ that we had before.

1. Radiation zone, power per unit solid angle

In radiation zone things are simpler as the magnetic field fixed by (83), in which we only act with the derivatives of a curl on the exponent e^{ikr} so that via (85),

$$\nabla \times \boldsymbol{p}_\omega \frac{e^{ikr}}{r} \approx ik\frac{e^{ikr}}{r}\boldsymbol{n} \times \boldsymbol{p}_\omega. \tag{121}$$

The same applies to the curl operation in (88) which now reads,

$$-i\omega\epsilon_0 \boldsymbol{E}_\omega = \nabla \times \boldsymbol{H}_\omega \tag{122}$$

This leads to the following expression for fields in the radiation zone,

$$\boldsymbol{H}_\omega(\boldsymbol{x}) = \frac{e^{ikr}}{4\pi r}\frac{\omega^2}{c}\boldsymbol{n} \times \boldsymbol{p}_\omega\,, \quad \boldsymbol{E}_\omega(\boldsymbol{x}) = \frac{e^{ikr}}{4\pi r}\frac{\omega^2}{\epsilon_0 c^2} [\boldsymbol{n} \times \boldsymbol{p}_\omega] \times \boldsymbol{n} \tag{123}$$

With the notations $\omega = ck$ and $Z_0 = \sqrt{\mu_0/\epsilon_0} = 1/c\epsilon_0$ (123) is (9.19) of J.D.J. Indeed, we have $\omega^2/c = ck^2$, and $[\frac{\omega^2}{\epsilon_0 c^2}]/[\frac{\omega^2}{c}] = 1/(c\epsilon_0) = Z_0$. Now the radiated power averaged over a period, according to (120), in dipole approximation

$$S = \boldsymbol{S} \cdot \boldsymbol{n} = \frac{1}{2}\mathrm{Re}\overline{\boldsymbol{E}_\omega \times \boldsymbol{H}_\omega^*} \cdot \boldsymbol{n} = \frac{1}{2}\frac{Z_0}{(4\pi r)^2}\frac{\omega^4}{c^2}\boldsymbol{n} \cdot \mathrm{Re}\left\{[(\boldsymbol{n} \times \boldsymbol{p}_\omega) \times \boldsymbol{n}] \times [\boldsymbol{n} \times \boldsymbol{p}_\omega^*]\right\} \tag{124}$$

$$\boldsymbol{n} \cdot \operatorname{Re} \{ [(\boldsymbol{n} \times \boldsymbol{p}_\omega) \times \boldsymbol{n}] \times [\boldsymbol{n} \times \boldsymbol{p}_\omega^*] \} = \boldsymbol{n} \cdot \operatorname{Re} \{ [-\boldsymbol{n}(\boldsymbol{n} \cdot \boldsymbol{p}_\omega) + \boldsymbol{p}_\omega] \times [\boldsymbol{n} \times \boldsymbol{p}_\omega^*] \} \tag{125}$$

And since

$$\boldsymbol{n} \cdot \operatorname{Re} \{ [\boldsymbol{n}(\boldsymbol{n} \cdot \boldsymbol{p}_\omega)] \times [\boldsymbol{n} \times \boldsymbol{p}_\omega^*] \} = \boldsymbol{n} \cdot \operatorname{Re} \{ (\boldsymbol{n} \cdot \boldsymbol{p}_\omega) [\boldsymbol{n}(\boldsymbol{n} \cdot \boldsymbol{p}_\omega^*) - \boldsymbol{p}_\omega^*] \} = 0, \tag{126}$$

and

$$\boldsymbol{n} \cdot \operatorname{Re} \{ [\boldsymbol{p}_\omega] \times [\boldsymbol{n} \times \boldsymbol{p}_\omega^*] \} = \boldsymbol{n} \cdot [\boldsymbol{n}|\boldsymbol{p}_\omega|^2 - \boldsymbol{p}_\omega^*(\boldsymbol{p}_\omega \cdot \boldsymbol{n})] = |\boldsymbol{p}_\omega|^2 - (\boldsymbol{p}_\omega^* \cdot \boldsymbol{n})(\boldsymbol{p}_\omega \cdot \boldsymbol{n}) \tag{127}$$

which is the same as

$$|(\boldsymbol{n} \times \boldsymbol{p}_\omega) \times \boldsymbol{p}_\omega|^2 = [\boldsymbol{p}_\omega - (\boldsymbol{p}_\omega \cdot \boldsymbol{n})\boldsymbol{n}] \cdot [\boldsymbol{p}_\omega^* - (\boldsymbol{p}_\omega^* \cdot \boldsymbol{n})\boldsymbol{n}] \tag{128}$$

we obtain, the expression for intensity averaged over a period as given by J.D.J.,

$$\frac{dP}{d\Omega} = S r^2 = \frac{Z_0}{32\pi^2} \frac{\omega^4}{c^2} |(\boldsymbol{n} \times \boldsymbol{p}_\omega) \times \boldsymbol{p}_\omega|^2 = \frac{c^2 Z_0 k^4}{32\pi^2} |(\boldsymbol{n} \times \boldsymbol{p}_\omega) \times \boldsymbol{p}_\omega|^2 = \frac{c^2 Z_0 k^4}{32\pi^2} \{ |\boldsymbol{p}_\omega|^2 - (\boldsymbol{p}_\omega^* \cdot \boldsymbol{n})(\boldsymbol{p}_\omega \cdot \boldsymbol{n}) \} \tag{129}$$

For the case when *all* the components of the complex amplitude of the dipole moment $\boldsymbol{p}_\omega$ have the same phase, the result (129) reduces to

$$\frac{dP}{d\Omega} = \frac{c^2 Z_0 k^4}{32\pi^2} |\boldsymbol{p}_\omega|^2 \sin^2 \theta \tag{130}$$

where according to the assumption of equal phase, α for all the components of $\boldsymbol{p}_\omega$ there exists a real vector $\boldsymbol{p}_\omega' = \boldsymbol{p}_\omega e^{-i\alpha}$, and θ is the angle between this vector $\boldsymbol{p}_\omega'$ and $\boldsymbol{n}$. Certainly in this case, $|\boldsymbol{p}_\omega|^2 = |\boldsymbol{p}_\omega'|^2 = (d_\omega'')^2$ where d_ω'' is the length of the real vector $\boldsymbol{p}_\omega'$. Equation (130) agrees with (101) as expected. Let's now reexamine the intensity of the radiation by the rotating dipole. In this case for the rotation in the positive sense and such that the dipole is aligned with x-axis at the time t_0, and with real and positive amplitude, d_0,

$$\boldsymbol{p}(t) = d_0[\hat{x} \cos(\omega(t - t_0)) + \hat{y} \sin(\omega(t - t_0))] = \operatorname{Re}[d_0(\hat{x} e^{-i\omega(t - t_0)} + i\hat{y} e^{-i\omega(t - t_0)})] = \operatorname{Re}[\boldsymbol{p}_\omega e^{-i\omega t}], \tag{131}$$

where the *complex* amplitude of the dipole moment reads,

$$\boldsymbol{p}_\omega = d_0 e^{i\omega t_0} (\hat{x} + i\hat{y}) \tag{132}$$

The overall phase is irrelevant, and we have in spherical coordinates,

$$|\boldsymbol{p}|^2 = d_0^2 (\hat{x} + i\hat{y}) \cdot (\hat{x} - i\hat{y}) = 2d_0^2, \qquad \boldsymbol{p}_\omega \cdot \boldsymbol{n} = d_0(\sin\theta\cos\phi + i\sin\theta\sin\phi) = d_0 \sin\theta e^{i\phi}, \tag{133}$$

Substitution of (133) in (129) yields for the averaged over a period intensity of radiation emitted by a rotating dipole,

$$\frac{dP}{d\Omega} = \frac{c^2 Z_0 k^4}{32\pi^2} d_0^2 (2 - \sin^2\theta) = \frac{c^2 Z_0 k^4}{32\pi^2} d_0^2 (1 + \cos^2\theta) \tag{134}$$

in agreement with (109) as it should.

2. Near (static) zone, $kr \ll 1$

Here instead of (121) we write,

$$\nabla \times \boldsymbol{p}_\omega \frac{e^{ikr}}{r} \approx -\frac{e^{ikr}}{r^2} \boldsymbol{n} \times \boldsymbol{p}_\omega. \tag{135}$$

and to find the electric field write

$$\nabla \times [\frac{e^{ikr}}{r^2} \boldsymbol{n} \times \boldsymbol{p}_\omega] \approx e^{ikr} \nabla \times [\frac{\boldsymbol{n}}{r^2} \times \boldsymbol{p}_\omega]. \tag{136}$$

By J.D.J vector identities, first page,

$$\nabla \times [\frac{\boldsymbol{n}}{r^2} \times \boldsymbol{p}_\omega] = -\boldsymbol{p}_\omega (\nabla \cdot \frac{\boldsymbol{n}}{r^2}) + (\boldsymbol{p}_\omega \cdot \nabla) \frac{\boldsymbol{n}}{r^2}. \tag{137}$$

Now the combination $\boldsymbol{n}/r^2$ is the electric field of the point charge located at the origin, $r = 0$. For that reason the first term of (136) vanishes as certainly $r \gg d$. The second term should be for the same reason the electric field of the dipole. Indeed,

$$(\boldsymbol{p}_\omega \cdot \nabla)\frac{\boldsymbol{n}}{r^2} = \frac{1}{r^3}\left(\boldsymbol{p}_\omega - 3\boldsymbol{n}(\boldsymbol{p}_\omega \cdot \boldsymbol{n})\right) \tag{138}$$

Combining (83), (118) and (135) we get

$$\boldsymbol{H} = \frac{i\omega}{4\pi r^2}\boldsymbol{n} \times \boldsymbol{p}_\omega \tag{139}$$

where $e^{ikr} \approx 1$ in near zone. So the magnetic field vanishes in the static limit $\omega \to 0$ and its time evolution is shifted by $\pi/2$ phase relative to the charges and currents oscillation. Now combining (122), (136), (138),(139) we find similarly

$$\boldsymbol{E} = \frac{1}{4\pi\epsilon_0 r^3}\left[3\boldsymbol{n}(\boldsymbol{n} \cdot \boldsymbol{p}_\omega) - \boldsymbol{p}_\omega\right] \tag{140}$$

This is the field of the dipole as if it is static at every moment. It is physically clear as the time it takes for electromagnetic disturbance to propagate from the radiation source is $T_{prop} \sim r/c$ is much smaller than the time over which the charges and current appreciably changes, $T_{change} \sim 1/\omega$. Indeed, $T_{prop}/T_{change} \sim r\omega/c = rk \ll 1$ in the near zone.

IV. ELECTRIC QUADRUPLE AND MAGNETIC DIPOLE APPROXIMATIONS

We expand (67) to the next (first) order in $kd \ll 1$. Lets limit discussion to the radiation zone, $kr \gg 1$, then $d/r \ll kd$ and we are allowed to stay at the zeroth order in d/r, namely start with (79). This limitation can be relaxed (see J.D.J.).

$$\boldsymbol{A}(x,t) = \frac{\mu_0}{4\pi r}\int d^3x' \boldsymbol{J}(\boldsymbol{x}', t - c^{-1}|\boldsymbol{x} - \boldsymbol{x}'|) \approx \frac{\mu_0}{4\pi r}\int d^3x' \boldsymbol{J}(\boldsymbol{x}', t - r/c) + \frac{\mu_0}{4\pi rc}\frac{\partial}{\partial t}\int d^3x'(r - |\boldsymbol{x} - \boldsymbol{x}'|)\boldsymbol{J}(\boldsymbol{x}', t - r/c) \tag{141}$$

and now to the leading order in d/r from (77),

$$\boldsymbol{A}(x,t) = \frac{\mu_0}{4\pi r}\int d^3x' \boldsymbol{J}(\boldsymbol{x}', t - c^{-1}|\boldsymbol{x} - \boldsymbol{x}'|) \approx \frac{\mu_0}{4\pi r}\int d^3x' \boldsymbol{J}(\boldsymbol{x}', t - r/c) + \frac{\mu_0}{4\pi rc}\frac{\partial}{\partial t}\int d^3x'(\boldsymbol{n}\boldsymbol{x}')\boldsymbol{J}(\boldsymbol{x}', t - r/c) \tag{142}$$

For the set of discreet charges (142) gives

$$\boldsymbol{A}(\boldsymbol{x},t) \approx \frac{\mu_0}{4\pi r}\sum_i e_i \dot{x}_i(t') + \frac{\mu_0}{4\pi rc}\frac{\partial}{\partial t}\sum_i e_i \dot{x}_i(t')(\boldsymbol{x}_i(t')\boldsymbol{n}), \quad t' \equiv t - r/c, \quad \partial_t = \partial_{t'} \tag{143}$$

By simple transformation with $\boldsymbol{r} \equiv \boldsymbol{x}_i$, $\boldsymbol{v} \equiv \dot{\boldsymbol{x}}_i$, (see L.L.)

$$\boldsymbol{v}(\boldsymbol{r} \cdot \boldsymbol{n}) = \frac{1}{2}\frac{\partial}{\partial t}\boldsymbol{r}(\boldsymbol{n} \cdot \boldsymbol{r}) + \frac{1}{2}\boldsymbol{v}(\boldsymbol{n} \cdot \boldsymbol{r}) - \frac{1}{2}\boldsymbol{r}(\boldsymbol{n} \cdot \boldsymbol{v}) = \frac{1}{2}\frac{\partial}{\partial t}\boldsymbol{r}(\boldsymbol{n} \cdot \boldsymbol{r}) + \frac{1}{2}[\boldsymbol{r} \times \boldsymbol{v}] \times \boldsymbol{n} \tag{144}$$

$$\boldsymbol{A}(x,t) = \frac{\mu_0}{4\pi r}\dot{\boldsymbol{p}}_{t'} + \frac{\mu_0}{8\pi rc}\frac{\partial^2}{\partial t^2}\sum_i e_i \boldsymbol{x}_i(t')(\boldsymbol{x}_i(t')\boldsymbol{n}) + \frac{\mu_0}{4\pi rc}\dot{\boldsymbol{m}} \times \boldsymbol{n} \tag{145}$$

$$\boldsymbol{m} = \frac{1}{2}\int d^3x' \boldsymbol{x}' \times \boldsymbol{j}(\boldsymbol{x}') = \sum_i \frac{e_i}{2}\boldsymbol{x}_i \times \dot{\boldsymbol{x}}_i \tag{146}$$

Addition to $\boldsymbol{A}$ of the function $\boldsymbol{n}\phi(r)$ doesn't change the magnetic field, as $\nabla \times \boldsymbol{n}\phi(r) = 0$. It in principle adds some longitudinal component to the electric field but it is of little concern as to find electric field from potentials we have

to consider also the scalar potential anyway. To avoid this complication we find the electric field from (88) instead. Therefore in (145) make a substitution,

$$\sum_i e_i \boldsymbol{x}_i(t')(\boldsymbol{x}_i(t')\boldsymbol{n}) \to \frac{1}{3}\sum_i \left\{ 3e_i\boldsymbol{x}_i(t')(\boldsymbol{x}_i(t')\boldsymbol{n}) - \boldsymbol{n}[\boldsymbol{x}_i(t')]^2 \right\} \tag{147}$$

quadruple moment tensor,

$$D_{\alpha\beta} = \sum_i e_i \left\{ 3x_i^\alpha x_i^\beta - \boldsymbol{x}^2 \delta_{\alpha\beta} \right\} \tag{148}$$

Vector $\boldsymbol{D}$,

$$\boldsymbol{D}_\alpha = D_{\alpha\beta}n_\beta \tag{149}$$

$$\sum_i e_i\boldsymbol{x}_i(t')(\boldsymbol{x}_i(t')\boldsymbol{n}) \to \boldsymbol{D}/3 \tag{150}$$

(145),

$$\boldsymbol{A}(x,t) = \frac{\mu_0}{4\pi r}\dot{\boldsymbol{p}} + \frac{\mu_0}{24\pi rc}\ddot{\boldsymbol{D}} + \frac{\mu_0}{4\pi rc}\dot{\boldsymbol{m}}\times\boldsymbol{n} \tag{151}$$

A. Radiation zone

Magnetic field,

$$\begin{aligned}
\boldsymbol{H}(x,t) &= \frac{1}{4\pi rc}\ddot{\boldsymbol{p}}\times\boldsymbol{n} + \frac{1}{24\pi rc^2}\dddot{\boldsymbol{D}}\times\boldsymbol{n} + \frac{1}{4\pi rc^2}[\ddot{\boldsymbol{m}}\times\boldsymbol{n}]\times\boldsymbol{n} \\
&= \frac{1}{4\pi rc}\left[\ddot{\boldsymbol{p}}\times\boldsymbol{n} + \frac{1}{6c}\dddot{\boldsymbol{D}}\times\boldsymbol{n} + \frac{1}{c}[\ddot{\boldsymbol{m}}\times\boldsymbol{n}]\times\boldsymbol{n}\right]
\end{aligned} \tag{152}$$

In the radiation zone, (88),(152) give

$$\boldsymbol{E} = Z_0\boldsymbol{H}\times\boldsymbol{n}\,, \quad Z_0 = \sqrt{\mu_0/\epsilon_0} \tag{153}$$

For harmonic fields $\boldsymbol{D}(t) = \boldsymbol{D}_\omega e^{-i\omega t}$ and the quadrupole contribution to the magnetic field,

$$\boldsymbol{H} = \frac{1}{24\pi rc^2}\dddot{\boldsymbol{D}}\times\boldsymbol{n} \tag{154}$$

can be rewritten using

$$\dddot{\boldsymbol{D}}(t') = (-i\omega)^3 e^{-i\omega t}e^{ikr}\boldsymbol{D}_\omega \tag{155}$$

as

$$\boldsymbol{H}_\omega = -\frac{ick^3}{24\pi}\frac{e^{ikr}}{r}\boldsymbol{n}\times\boldsymbol{D}_\omega \tag{156}$$

(156) is (9.44) of JDJ. Intensity of quadrupole radiation, from (32), (153), (154)

$$\frac{dP}{d\Omega} = r^2 Z_0 \left(\frac{1}{24\pi rc^2}\right)^2 (\dddot{\boldsymbol{D}}\times\boldsymbol{n})^2 \tag{157}$$

The direction dependence of (157) is in general complicated, but with the definition (149) the total intensity is simpler.

$$(\dddot{\boldsymbol{D}}\times\boldsymbol{n})^2 = (\dddot{\boldsymbol{D}})^2 - (\dddot{\boldsymbol{D}}\cdot\boldsymbol{n})^2 = \dddot{D}_{\alpha,\beta}\dddot{D}_{\alpha,\gamma}n_\beta n_\gamma - \dddot{D}_{\alpha,\beta}\dddot{D}_{\gamma,\delta}n_\alpha n_\beta n_\gamma n_\delta \tag{158}$$

Using relations

$$\langle n_\beta n_\gamma\rangle = \frac{\delta_{\beta\gamma}}{3} \quad \langle n_\alpha n_\beta n_\gamma n_\delta\rangle = \frac{1}{15}(\delta_{\alpha\beta}\delta_{\gamma\delta} + \delta_{\alpha\gamma}\delta_{\beta\delta} + \delta_{\alpha\delta}\delta_{\beta\gamma}) \tag{159}$$

Therefore,

$$\oint d\Omega(\dddot{\boldsymbol{D}} \times \boldsymbol{n})^2 = \frac{4\pi}{3}\dddot{D}_{\alpha,\beta}\dddot{D}_{\alpha,\beta} - \frac{4\pi}{15}(\dddot{D}_{\alpha,\alpha}\dddot{D}_{\beta,\beta} + \dddot{D}_{\alpha,\beta}\dddot{D}_{\alpha,\beta} + \dddot{D}_{\alpha,\beta}\dddot{D}_{\beta,\alpha}) = \frac{4\pi}{5}\dddot{D}_{\alpha,\beta}^2, \tag{160}$$

where we used the fact that the tensor D is symmetric and traceless, and introduced the notation, $\dddot{D}_{\alpha,\beta}\dddot{D}_{\alpha,\beta} \equiv \dddot{D}_{\alpha,\beta}^2$. Therefore, the total radiated power is

$$P = Z_0\left(\frac{1}{24\pi c^2}\right)^2\frac{4\pi}{5}\dddot{D}_{\alpha,\beta}^2 = \frac{Z_0}{720\pi c^4}\dddot{D}_{\alpha,\beta}^2. \tag{161}$$

Note the extra factor of $1/2$ in JDJ always added for the Poynting vector with harmonic fields.

B. Example of a quadrupole radiation

Carbon dioxide vibrational mode, CO_2, has a configuration, $O = C = O$ Let the charge at C be $-2q$ and the two charges at O be q. Let the molecule be aligned along z-axis and assume the two charges q perform the harmonic motion, $z_{1,2}(t) = \pm(a + b\cos\omega t)$, then the quadrupole moment is

$$D_{\alpha\beta} = D_0\begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{bmatrix}, \quad D_0 = 2q(a + b\cos\omega t)^2 \tag{162}$$

where the factor of 2 accounts for two charges at two oxygen atoms.

$$\dddot{D}_0(t_r) = 4qb\omega^3(a\sin\omega t_r + 2b\sin 2\omega t_r), \quad t_r = t - r/c \tag{163}$$

By (149), (A1) the vector,

$$\boldsymbol{D} = \dddot{D}_0(t_r)(-\hat{x}\sin\theta\cos\phi - \hat{y}\sin\theta\sin\phi + 2\hat{z}\cos\theta) \tag{164}$$

and correspondingly by (A6), (A7)

$$\dddot{\boldsymbol{D}} \times \boldsymbol{n} = 3\dddot{D}_0\hat{\phi}\sin\theta\cos\theta \tag{165}$$

Note the appearance of the factor of 3. Then (154) gives

$$\boldsymbol{H} = \hat{\phi}\frac{\dddot{D}_0}{8\pi rc^2}\sin\theta\cos\theta \tag{166}$$

and by (153) and (A2)

$$\boldsymbol{E} = \hat{\theta}Z_0\frac{\dddot{D}_0}{8\pi rc^2}\sin\theta\cos\theta \tag{167}$$

For $a = 0$ the radiation is linearly polarized and monochromatic at the frequency 2ω. The radiated power is by (32),

$$\frac{dP}{d\Omega} = Z_0\left(\frac{\dddot{D}_0}{8\pi c^2}\right)^2\sin^2\theta\cos^2\theta \tag{168}$$

has a typical quadrupole angular dependence, $\propto \sin^2\theta\cos^2\theta$. Averaged over the radiation period, $\overline{(\text{note }\sin^2(\omega t)} = 1/2$ the same factor of $1/2$ must be included explicitly in the harmonic fields formalism), Also note that $\overline{\sin(\omega t)\sin(2\omega t)} = 0$ so no terms $\propto ab$,

$$\overline{\frac{dP}{d\Omega}} = Z_0\frac{1}{2}\left(\frac{1}{8\pi c^2}\right)^2\sin^2\theta\cos^2\theta(4qb\omega^3)^2(a^2 + 4b^2) \tag{169}$$

This radiation power scales as ω^6 unlike the fourth power typical for the dipole radiation.

C.   Example of the magnetic dipole

Consider the current loop in $x - y$ plane concentric with the origin of a radius R with the current, $I(t) = I_0 \cos(\omega t)$ ($\int dS$ is integration over the cross-section of the loop.

$$\boldsymbol{m}(t) = \oint dl \int dS \frac{e}{2} (\boldsymbol{x} \times \boldsymbol{v}) \cdot \boldsymbol{n} = \oint dl \frac{x}{2} \times \boldsymbol{I}(t) = \hat{z} \frac{1}{2} R I(t) 2\pi R = \hat{z} I_0 \cos(\omega t) \pi R^2 = \hat{z} m_0 \cos(\omega t) \tag{170}$$

The vector potential (151)

$$\boldsymbol{A} = \frac{\mu_0}{4\pi r c} \dot{\boldsymbol{m}} \times \boldsymbol{n} = \frac{\mu_0}{4\pi r c} \pi R^2 I_0(-\omega) \sin(\omega(t - r/c)) \boldsymbol{z} \times \boldsymbol{n} = -\frac{\mu_0 m_0 \omega}{4\pi c} \frac{\sin\theta}{r} \hat{\phi} \sin(\omega(t - r/c)) \tag{171}$$

We actually do not need it since from (152)

$$\boldsymbol{H} = \frac{1}{4\pi r c^2} (\ddot{\boldsymbol{m}} \times \boldsymbol{n}) \times \boldsymbol{n} \tag{172}$$

using (A2) and (A6)

$$\ddot{\boldsymbol{m}} = -\omega^2 m_0 \cos(\omega t)\hat{z}, \quad (\hat{z} \times \boldsymbol{n}) \times \boldsymbol{n} = \hat{\phi} \sin\theta \times \boldsymbol{n} = \sin\theta \hat{\theta} \tag{173}$$

we get

$$\boldsymbol{H} = -\frac{m_0 \omega^2}{4\pi c^2} \left(\frac{\sin\theta}{r} \right) \cos(\omega t_r)\hat{\theta} \tag{174}$$

and from the radiation zone result, Eq. (153),

$$\boldsymbol{E} = +Z_0 \frac{m_0 \omega^2}{4\pi c^2} \left(\frac{\sin\theta}{r} \right) \cos(\omega t_r)\hat{\phi} \tag{175}$$

Averaged radiated power, (95),

$$\frac{dP}{d\Omega} = Z_0 \frac{1}{2} \left(\frac{m_0 \omega^2}{4\pi c^2} \right)^2 \sin^2\theta \tag{176}$$

V.   ESTIMATES OF RELATIVE IMPORTANCE OF DIFFERENT CONTRIBUTIONS

To compare magnetic dipole to the electric dipole contribution note from the above examples that

$$P_{mag,dipole} \propto Z_0 m_0^2 k^4 \tag{177}$$

and the electric dipole

$$P_{elec,dipole} \propto Z_0 c^2 k^4 p^2 \tag{178}$$

so the ratio is typically,

$$\frac{P_{mag,dipole}}{P_{elec,dipole}} \propto \frac{m_0^2}{c^2 p^2} \tag{179}$$

Imagine the radiation is due to the point charge q moving along the circular orbit. Then,

$$m_0 \propto qvR, \quad p \propto qR \tag{180}$$

and

$$\frac{m_0^2}{c^2 p^2} \propto \left(\frac{qvR}{cqR} \right)^2 = \frac{v^2}{c^2} \ll 1 \tag{181}$$

which is indeed an expansion parameter squared which was assumed to be small.

Let us estimate the contribution of the quadrupole. From the above examples,

$$P_{quadr} \propto Z_0 \frac{q^2 R^4 \omega^6}{c^4} = Z_0 q^2 R^4 k^6 c^2 \tag{182}$$

So that

$$\frac{P_{quadr}}{P_{dipole}} \propto \frac{q^2 R^4 k^6 c^2}{c^2 k^4 p^2} \propto R^2 k^2 \approx (R/\lambda)^2 \propto (v/c)^2 \tag{183}$$

which is the same smallness as in the case of a magnetic dipole contribution.

VI.   LARMOR FORMULA AND CYCLOTRON RADIATION

Consider the *non*-relativistic point-like charge q moving along the trajectory, $\boldsymbol{r}(t)$. The dipole moment is $\boldsymbol{p} = q\boldsymbol{r}(t)$ and $\ddot{\boldsymbol{p}} = q\ddot{\boldsymbol{r}}(t) = \boldsymbol{a}(t)$. Substitute it to (92) to obtain in the radiation zone,

$$\boldsymbol{E} = \frac{1}{4\pi\epsilon_0 r c^2} \boldsymbol{n} \times (\boldsymbol{n} \times \ddot{\boldsymbol{p}}) = \frac{\mu_0 q}{4\pi r} \boldsymbol{n} \times (\boldsymbol{n} \times \boldsymbol{a}(t - r/c)) \tag{184}$$

as a result we get the Larmor relation

$$dP/d\Omega = \frac{\mu_0 q^2 |\boldsymbol{a}_{ret}|^2}{16\pi^2 c} \sin^2 \theta \tag{185}$$

For the cyclotron radiation just substitute

$$a = v^2/R = v^2/(mv/qB) \tag{186}$$

to get

$$P = \frac{q^4 v^2 B^2}{6\pi\epsilon_0 m^2 c^3} \tag{187}$$

Appendix A: Polar coordinates

$$\boldsymbol{n} = (n_1, n_2, n_3) = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta) \tag{A1}$$

At each direction $\hat{n}$ triple of mutually orthogonal vectors is defined $\hat{\theta}, \hat{\phi}, \hat{n}$ such that

$$\hat{n} \times \hat{\theta} = \hat{\phi}, \quad \hat{\theta} \times \hat{\phi} = \hat{n}, \quad \hat{\phi} \times \hat{n} = \hat{\theta}. \tag{A2}$$

$$\hat{z} = \hat{n}\cos\theta - \hat{\theta}\sin\theta \tag{A3}$$

$$\hat{x} = -\hat{\phi}\sin\phi + \hat{\theta}\cos\phi\cos\theta + \hat{n}\sin\theta\cos\phi \tag{A4}$$

$$\cos\alpha\hat{x} + \sin\alpha\hat{y} = \hat{\theta}\sin\theta\cos(\phi - \alpha) + \hat{n}\cos\theta\cos(\phi - \alpha) - \hat{\phi}\sin(\phi - \alpha) \tag{A5}$$

In studies of radiation it is useful to have vector products,

$$\hat{n} \times \hat{z} = -\hat{\phi}\sin\theta \tag{A6}$$

$$\hat{n} \times [\cos\alpha\hat{x} + \sin\alpha\hat{y}] = \hat{\phi}\cos\theta\cos(\phi - \alpha) + \hat{\theta}\sin(\phi - \alpha) \tag{A7}$$

Appendix B: Vector Analysis Identities

For a constant vector $\boldsymbol{C}$

$$\nabla \times [\boldsymbol{C} \times \boldsymbol{f}(\boldsymbol{x})] = \boldsymbol{C}[\nabla \cdot \boldsymbol{f}(\boldsymbol{r})] - (\boldsymbol{C} \cdot \nabla)\boldsymbol{f}(\boldsymbol{x}) \tag{B1}$$