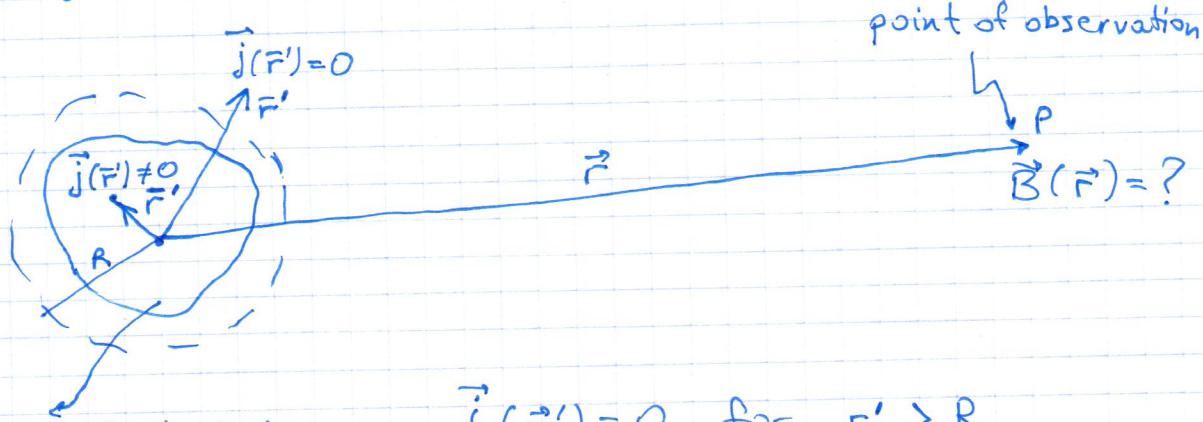


# Ch. 11 (A.Z.) Magnetic Multipoles

We study the magnetic field,  $\vec{B}$  produced by the current distribution  $\vec{j}(\vec{r}')$  limited to a finite volume in space, at an observation point  $\vec{r}$  far from the distribution



current distribution,  $\vec{j}(\vec{r}') = 0$  for  $r' > R$

What is the field  $\vec{B}(\vec{r})$  for  $r \gg R$  to the leading order in  $R/r$ ? The current distribution is assumed to be stationary:  $\vec{\nabla}' \cdot \vec{j}(\vec{r}') = 0$

We consider the vector potential,  $\vec{A}(\vec{r})$  in a Coulomb gauge,  $\vec{\nabla} \cdot \vec{A} = 0$ , where it is given by

$$\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi} \int d^3 r' \frac{\vec{j}(\vec{r}')}{|\vec{r} - \vec{r}'|} \quad (1)$$

As  $r' \ll r$ , make an expansion

$$(\vec{r} - \vec{r}')^{-1} = \frac{1}{r} + \frac{\vec{r}' \cdot \vec{r}}{r^3} + \dots \quad (2)$$

$$A_k(\vec{r}) = \frac{\mu_0}{4\pi} \left[ \underbrace{\frac{1}{r} \int d^3 r' \delta_k(\vec{r}')}_{\text{magnetic "monopole"}} + \underbrace{\frac{\vec{r}}{r^3} \cdot \int d^3 r' \vec{j}(\vec{r}') \vec{r}' + \dots}_{\text{magnetic dipole}} \right] \quad (3)$$

### 11.1.2 The magnetic monopole

The first term in the expansion vanishes

This has nothing to do with the absence of magnetic charge in nature, but follow from the stationarity condition,  $\vec{\nabla}' \cdot \vec{J}(\vec{r}') = 0$

$$\vec{\nabla}' \cdot (r_k' \vec{j}(\vec{r}')) = r_k' \underbrace{\vec{\nabla}' \cdot \vec{J}(\vec{r}')}_{\delta} + \vec{j} \cdot \underbrace{\vec{\nabla}' r_k'}_{\text{unit vector along } k\text{-th axis}} = j_k$$

$$\Rightarrow \int d^3 r' j_k(\vec{r}') = \int d^3 r' \vec{\nabla}' \cdot (r_k' \vec{j}(\vec{r})) = \oint d\vec{s}' (r_k' \vec{j}) = 0$$

since the current distribution is localized in space.

The meaning of the statement  $\int d^3 r' \vec{j}(\vec{r}') = 0$  is especially clear for filamentary current loop,  $I \oint d\vec{l} = 0$

### 11.2 The magnetic dipole

We've shown that the first term of Eq.(3)

vanishes. If the second term is non-zero, the asymptotic behavior is

$$A_k(\vec{r}) = \frac{\mu_0}{4\pi} \left[ \int d^3 r' j_k(\vec{r}') r_e' \right] \frac{r_e}{r^3} = \frac{\mu_0}{4\pi} T_{k\ell e} \frac{r_e}{r^3}, \quad (5)$$

where the repeated index is summed over

We are going to simplify Eq.(5) using two identities,

1-st identity:

$$\begin{aligned} \vec{\nabla}' \cdot (r_e' r_k' \vec{j}) &= r_e' r_k' \underbrace{\vec{\nabla}' \cdot \vec{j}}_{\delta} + r_e' j_k + r_k' j_e = \\ &= r_e' j_k + r_k' j_e \end{aligned} \quad (6)$$

2-nd identity

$$\epsilon_{ek\dot{i}} (\vec{r}' \times \vec{j})_i = r'_e j_k - r'_k j_e$$

indeed,  $\epsilon_{ek\dot{i}} (\vec{r}' \times \vec{j})_i = \epsilon_{ek\dot{i}} \epsilon_{ist} r'_s j_t =$

$$= (\delta_{es} \delta_{kt} - \delta_{et} \delta_{ks}) r'_s j_t = r'_e j_k - r'_k j_e \quad (7)$$

Let's add Eqs.(6) and (7), and get

$$j_k r'_e = \frac{1}{2} \vec{\nabla}' \cdot (r'_e r'_k \vec{j}) + \frac{1}{2} \epsilon_{ek\dot{i}} (\vec{r}' \times \vec{j})_i$$

$\Rightarrow$  By definition, Eq. (5)

$$T_{ke} = \int d^3 r' j_k(\vec{r}') r'_e = \frac{1}{2} \int d^3 r' \vec{\nabla}' \cdot (r'_e r'_k \vec{j}) + \frac{1}{2} \epsilon_{ek\dot{i}} \int d^3 r' (\vec{r}' \times \vec{j})_i \quad (8)$$

The first term vanishes for localized current distribution according to the divergence theorem.

The second term can be rewritten using the definition of the magnetic moment of a current distribution:

$$\vec{m} = \frac{1}{2} \int d^3 \vec{r}' \vec{r}' \times \vec{j}(\vec{r}') \quad (9)$$

$$Eq.(9) \Rightarrow Eq.(8) \Rightarrow T_{ke} = \epsilon_{kie} m_i \quad (9) A$$

$$Eq.(5) \Rightarrow A_k(\vec{r}) = \frac{\mu_0}{4\pi} T_{ke} \frac{r_e}{r^3} = \frac{\mu_0}{4\pi} \epsilon_{kie} m_i \frac{r_e}{r^3} \Rightarrow$$

$$\Rightarrow \boxed{\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi} \frac{\vec{m} \times \vec{r}}{r^3}} \quad r \gg R \quad (10)$$

Note  $\vec{m}$  is independent of a point of origin:

$$\vec{m}' = \frac{1}{2} \int d^3 \vec{r}' \vec{r}' \times \vec{j}(\vec{r}' - \vec{a}) = \frac{1}{2} \int d^3 \vec{r}' (\vec{r}' + \vec{a}) \times \vec{j}(\vec{r}') = \frac{1}{2} \int d^3 \vec{r}' \vec{r}' \times \vec{j}(\vec{r}') \equiv \vec{m}$$

Let's check Eq.(9) and Eq.(10) in the case of

a circular loop with current I (radius  $a$ ).

It's convenient to choose the origin at the center of the loop, then

$$\vec{m} = \frac{1}{2} \oint d\ell' \vec{r}' \times \vec{I}(\ell') = \hat{z} \frac{1}{2} 2\pi a (aI) = \hat{z} \pi a^2 I$$

$$\Rightarrow \vec{A} = \frac{\mu_0}{4\pi} \pi a^2 I \hat{z} \times \frac{1}{r^3} = \frac{\mu_0}{4\pi} \pi a^2 I \frac{\hat{z}}{r^2} \sin\theta \quad (11)$$

which is indeed what we've got by direct evaluation of the vector potential of a loop previously.

Let's now find the dipolar magnetic field,  $\vec{B}(r)$ ,

$$\vec{B}(r) = \vec{\nabla} \times \vec{A}(r) \quad (12)$$

Use the identity (see Appendix) valid for any constant vector  $\vec{C}$  and any vector function,  $\vec{f}(r)$ ,

$$\vec{\nabla} \times (\vec{C} \times \vec{f}(r)) = \vec{C}(\vec{\nabla} \cdot \vec{f}) - (\vec{C} \cdot \vec{\nabla})\vec{f} \quad (13)$$

to write

$$\vec{B}(r) = \vec{\nabla} \times \left( \frac{\mu_0}{4\pi} \frac{\vec{m} \times \vec{r}}{r^3} \right) = \frac{\mu_0}{4\pi} \left\{ \vec{m} \left( \vec{\nabla} \cdot \frac{\vec{r}}{r^3} \right) - (\vec{m} \cdot \vec{\nabla}) \frac{\vec{r}}{r^3} \right\} \quad (14)$$

Consider  $r \neq 0$  (otherwise  $\delta$ -function arises),

$\vec{\nabla} \cdot \left( \frac{\vec{r}}{r^3} \right) = 0$ , as  $\frac{\vec{r}}{r^3}$  is proportional to the electric field of a point-charge at origin.

$$\Rightarrow \vec{B}(r) = - \frac{\mu_0}{4\pi} (\vec{m} \cdot \vec{\nabla}) \frac{\vec{r}}{r^3} = \frac{\mu_0}{4\pi} \frac{3\vec{r}(\vec{r} \cdot \vec{m}) - \vec{m}}{r^3} \quad (r \neq 0) \quad (15)$$

Let's demonstrate the last equality

Use the following relations (Appendix)

$$(1) \partial_k r_e = \delta_{k,e}, (2) \partial_k |\vec{r}| = (\hat{\vec{r}})_k, k\text{-th component of } \hat{\vec{r}} = \frac{\vec{r}}{r} = \frac{\vec{r}_k}{r}$$

then

$$\partial_k \left( \frac{r_e}{r^3} \right) = \frac{1}{r^6} \left( (\partial_k r_e) \cdot r^3 - r_e \partial_k r^3 \right) =$$

$$= \frac{1}{r^6} \left( \delta_{k,e} r^3 - r_e 3r^2 (\hat{\vec{r}})_k \right) = \frac{1}{r^3} \left( \delta_{k,e} - 3(\hat{\vec{r}})_e (\hat{\vec{r}})_k \right)$$

as a result

$$B_e = - \frac{\mu_0}{4\pi} (\vec{m} \cdot \vec{\nabla}) \frac{\vec{r}_e}{r^3} = - \frac{\mu_0}{4\pi} m_k \partial_k \frac{r_e}{r^3} =$$

$$= - \frac{\mu_0}{4\pi} m_k \frac{1}{r^3} (\delta_{k,e} - 3(\hat{\vec{r}})_e (\hat{\vec{r}})_k) =$$

$$= - \frac{\mu_0}{4\pi} \frac{m_e - 3(\hat{\vec{r}})_e (\hat{\vec{r}}) \cdot \vec{m}}{r^3} \Rightarrow \vec{B}(\vec{r}) = \frac{\mu_0}{4\pi} \frac{3\hat{\vec{r}}(\vec{m} \cdot \hat{\vec{r}}) - \vec{m}}{r^3},$$

which proves Eq. (15).

The following identity is useful: (see Appendix)

$$(\vec{m} \cdot \vec{\nabla}) \frac{\vec{r}}{r^3} = \vec{\nabla} \left( \frac{\vec{m} \cdot \vec{r}}{r^3} \right), \text{ for } \underline{r \neq 0}!$$

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Eq. (16) allows us to rewrite Eq. (14), (15) in a form

$$\vec{B}(\vec{r}) = - \frac{\mu_0}{4\pi} \vec{\nabla} \left( \frac{\vec{m} \cdot \vec{r}}{r^3} \right), \text{ for } \underline{r \neq 0}, \text{ which is} \\ \text{okay, as anyway } r \gg R.$$

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from Eq. (17)  
It follows  $\checkmark$  that the dipolar field is proportional

to the field of a dipole (formally exchange

$\vec{m} \rightarrow \vec{P}$ ,  $\mu_0 \rightarrow \frac{1}{\epsilon_0}$  to obtain dipole electric field

from Eq. (17))

### 11.2.1 The magnetic dipole moment of a filamentary current (planar case)



this contributes

$$\vec{m} = \frac{1}{2} \int d^3r \vec{F} \times \vec{j}(\vec{r}) \rightarrow$$

$$\rightarrow \frac{I}{2} \oint \vec{F} \times d\vec{l} \Rightarrow$$

$$I \propto \frac{1}{2} \vec{F} \times d\vec{l} = d\vec{S} \parallel \odot \Rightarrow \vec{m} = I \vec{S} \text{ (Fig.)}$$

i.e.  $\vec{m}$  is a directed area of a loop,  $\vec{S}$  times current,  $I$ .

### 11.2.2. Orbital and Spin Magnetic Moments

$$\vec{S}(\vec{r}) = \sum_{k=1}^N q_k \vec{v}_k \delta(\vec{r} - \vec{r}_k) \leftarrow \begin{array}{l} \text{orbital} \\ \text{current} \end{array}$$

charge  $q_k$   $\vec{v}_k$  the orbital magnetic moment

$N$  particles in motion

$$\vec{m}_L = \frac{1}{2} \int d^3r' \vec{F}' \times \vec{j}(\vec{r}') =$$

$$= \frac{1}{2} \sum_{k=1}^N q_k (\vec{r}_k \times \vec{v}_k) = \sum_{k=1}^N \frac{q_k}{2m_k} \vec{L}_k,$$

where  $\vec{L}_k = \vec{r}_k \times m_k \vec{v}_k$  is the angular momentum of  $k$ -th particle.

Suppose all the particles have the same charge-to-mass ratio,  $q/m$ . Then

$$\vec{m}_L = \frac{q}{2m} \vec{L}, \text{ where } \vec{L} = \sum_k \vec{L}_k \text{ is the total angular momentum}$$

The relation

$$\boxed{\vec{m} = \frac{q}{2m} \vec{L}}$$

(18)

is true quantum-mechanically as well

Spin magnetic moment of a particle,  $\vec{m}_s = g \frac{q}{2m} \vec{S}$

where  $\vec{S}$  is the spin angular momentum is responsible for most of the magnetic phenomena in solids.

For electrons  $g = 2 + \text{small corrections}$

Dirac Equation

↑ QED interaction with EM field

(radiative corrections)

In generic situation  $\vec{m} = \gamma \vec{J}$ ,

where  $\vec{J}$  is the total angular momentum, and  $\gamma$  is gyromagnetic ratio.

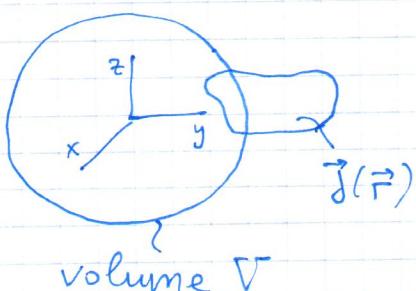
### Example 11.1 (A.Z.)

Let  $\vec{B}(r)$  be the magnetic field produced by a current inside a spherical volume  $V$  of radius  $R$ . Show that the magnetic moment of  $\vec{j}(r)$  is

$$\vec{m} = \frac{3}{2\mu_0} \int_V d^3r \vec{B}(r)$$

Solution First let  $\vec{j}(r)$  not be enclosed by  $V$ .

Place the origin at the center of  $V$ .



$$\begin{aligned} \frac{1}{V} \int_V d^3r \vec{B}(r) &= \\ &= -\frac{\mu_0}{4\pi} \int_V d^3r' \vec{j}(r') \times \int_V d^3r \frac{\vec{r}' - \vec{r}}{|\vec{r}' - \vec{r}|^3} \end{aligned}$$

Note that the integral  $\int_V d^3r \frac{\vec{F}' - \vec{F}}{|F' - F|^3}$  is

(P.8)

an electric field at  $\vec{r}'$ ,  $\vec{E}(\vec{r}')$  produced by a uniform charge density  $\rho(\vec{r}) = 4\pi\epsilon_0 \equiv \rho$  confined to a volume  $V$ .

Since  $V$  is a sphere of radius  $R$ , Gauss law gives

$$\vec{E}(\vec{r}') = \begin{cases} \frac{\rho \vec{r}'}{3\epsilon_0} = \frac{4\pi}{3} \vec{r}' & , r' < R \\ \frac{\rho V}{4\pi\epsilon_0 r'^3} \vec{r}' = V \frac{\vec{r}'}{r'^3} & , r' > R \end{cases}$$

$$\Rightarrow \frac{1}{V} \int_V d^3r \vec{B}(\vec{r}) = \frac{\mu_0}{3V} \int_{r' < R} d^3r' \vec{r}' \times \vec{j}(\vec{r}') - \frac{\mu_0}{4\pi} \int_{r' > R} d^3r' \frac{\vec{j}(\vec{r}') \times \vec{r}'}{r'^3} = \\ = \frac{2}{3} \mu_0 \frac{\vec{m}_{in}}{V} + \vec{B}_{out}(0) , \text{ where}$$

$\vec{m}_{in}$  is the magnetic dipole moment due to currents inside  $V$ ,  $\vec{B}_{out}(0)$  is the field due to currents outside  $V$  at the origin.

Two cases:

A) All the currents are inside  $V$ ,

then  $\vec{B}_{out}(0) = 0$ ,  $\vec{m}_{in}$  becomes  $\vec{m}$ , a total magnetic moment of all currents,

$$\boxed{\int_V d^3r \vec{B}(\vec{r}) = \frac{2\mu_0}{3} \vec{m}}$$

(19)

B) No currents inside  $V$ , then  $\vec{m}_{in} = 0$ ,

and  $\vec{B}_{out}$  becomes  $\vec{B}$ , a total field due to all currents,  $\vec{B}(0) = \frac{1}{V} \int_V d^3r \vec{B}(\vec{r})$ , "mean value theorem"

### 11.2.3 The point Magnetic Dipole.

p.9

Electron is such an object without approximation  
(seems, there are no point electric dipoles in nature)

The vector potential using Eq. (10), now without restriction  $r \gg R$  gives

$$\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi} \frac{\vec{m} \times (\vec{r} - \vec{r}_0)}{|\vec{r} - \vec{r}_0|^3}, \text{ where}$$

$\vec{r}_0$  is the location of a dipole  $\vec{m}$

Let's evaluate  $\vec{B}(\vec{r}) = \vec{\nabla} \times \vec{A}(\vec{r})$  using (Appendix)

$$\vec{\nabla} \times (\vec{C} \times \vec{f}(\vec{r})) = \vec{C} (\vec{\nabla} \cdot \vec{f}) - (\vec{C} \cdot \vec{\nabla}) \vec{f},$$

$$\text{and } \frac{\vec{r} - \vec{r}_0}{|\vec{r} - \vec{r}_0|^3} = -\vec{\nabla} \frac{1}{|\vec{r} - \vec{r}_0|}$$

$$\vec{\nabla} \times \vec{A} = \frac{\mu_0}{4\pi} \left[ \vec{m} \underbrace{\left( \vec{\nabla}^2 \frac{(-1)}{|\vec{r} - \vec{r}_0|} \right)}_{4\pi \delta(\vec{r} - \vec{r}_0)} - (\vec{m} \cdot \vec{\nabla}) \frac{\vec{r} - \vec{r}_0}{|\vec{r} - \vec{r}_0|^3} \right]$$
(20)

which can be written using Eq. (16) as

$$\vec{\nabla} \times \vec{A} = \mu_0 \left[ \vec{m} \delta(\vec{r} - \vec{r}_0) - \vec{\nabla} \frac{1}{4\pi} \frac{\vec{m} \cdot (\vec{r} - \vec{r}_0)}{|\vec{r} - \vec{r}_0|^3} \right]$$
(21)

which allows to identify the dipolar current,

$$\vec{j} = \frac{1}{\mu_0} \vec{\nabla} \times \vec{B} \Rightarrow \vec{j}(\vec{r}) = \vec{\nabla} \times [\vec{m} \delta(\vec{r} - \vec{r}_0)] = -\vec{m} \times \vec{\nabla} \delta(\vec{r} - \vec{r}_0),$$

Where we have used identity (Appendix)

$$\vec{\nabla} \times [\vec{C} f(\vec{r})] = -\vec{C} \times \vec{\nabla} f(\vec{r})$$

However, the second term of Eq.(21) contains  
a hidden  $\delta$ -function. On the other hand  
we know that for  $\vec{r} \neq \vec{r}_0$ ,

$$\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi} \left[ \frac{3\hat{n}(\hat{n} \cdot \vec{m}) - \vec{m}}{|\vec{r} - \vec{r}_0|^3} \right], \quad \vec{r} \neq \vec{r}_0 \quad (22)$$

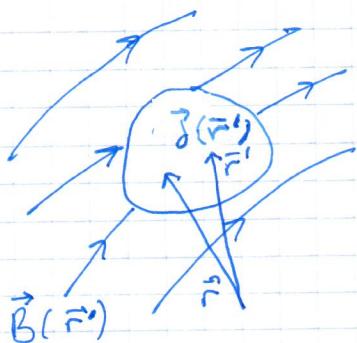
where  $\hat{n} = \frac{\vec{r} - \vec{r}_0}{|\vec{r} - \vec{r}_0|}$ . To obtain the expression valid for all  $\vec{r}$ , including  $\vec{r} = \vec{r}_0$  we have to add to Eq.(22) the term  $\propto \delta(\vec{r} - \vec{r}_0)$  with the proper coefficient. This coefficient can be fixed, e.g. by the requirement, Eq.(19),

$$\int d^3r \vec{B}(\vec{r}) = \frac{2\mu_0}{3} \vec{m}, \text{ which results in}$$

$$\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi} \left[ \frac{3\hat{n}(\hat{n} \cdot \vec{m}) - \vec{m}}{|\vec{r} - \vec{r}_0|^3} + \frac{8\pi}{3} \vec{m} \delta(\vec{r} - \vec{r}_0) \right], \text{ all } \vec{r} \quad (23)$$

Expression (23) satisfies, Eq.(19) since the angular integral of the first term over the directions of  $\hat{n}$  vanishes.

#### 12.4.1 The Dipole Force



Given the current distribution  $\vec{J}(\vec{r}')$  find the force acting on  $\vec{J}(\vec{r}')$  when the latter is placed in a magnetic field,  $\vec{B}(\vec{r}')$

$$\vec{F} = \int d^3r' \vec{J}(\vec{r}') \times \vec{B}(\vec{r}') \quad (25)$$

Make an expansion  $\vec{B}(\vec{r}') = \vec{B}(\vec{r}) + [(\vec{r}' - \vec{r}) \cdot \vec{\nabla}] \vec{B}(\vec{r}) + \dots$  (26)

Where  $\vec{r}$  is the reference point inside the

distribution  $\vec{\delta}(\vec{r}')$ . The expansion makes sense if  $\vec{B}(\vec{r}')$  changes slowly on a scale of a size of distribution  $\vec{j}(\vec{r}')$ .

(26)  $\Rightarrow$  (25) gives

$$\begin{aligned}\vec{F} &= \int d^3 r' \vec{j}(\vec{r}') \times [\vec{B}(\vec{r}) + [(r' - \vec{r}) \cdot \vec{\nabla}] \vec{B}(\vec{r}) + \dots] = \\ &= \underbrace{\left[ \int d^3 r' \vec{\delta}(\vec{r}') \right]}_{\text{earlier was shown to vanish}} \times \vec{B}(\vec{r}) + \int d^3 r' \vec{j}(\vec{r}') \times [(r' - \vec{r}) \cdot \vec{\nabla}] \vec{B}(\vec{r})\end{aligned}$$

The second term contains two pieces

$$\begin{aligned}\vec{F} &= \underbrace{\left[ \int d^3 r' \vec{\delta}(\vec{r}') \right]}_{\text{this again is zero}} \times (-\vec{r} \cdot \vec{\nabla}) \vec{B}(\vec{r}) + \int d^3 r' \vec{j}(\vec{r}') \times [(\vec{r}' \cdot \vec{\nabla}) \vec{B}(\vec{r})] \\ &\Rightarrow \vec{F} = \int d^3 r' \vec{j}(\vec{r}') \times [(\vec{r}' \cdot \vec{\nabla}) \vec{B}(\vec{r})] \\ &\Rightarrow F_s = \int d^3 r' \epsilon_{\text{stm}} \delta_t(\vec{r}') r'_e \frac{\partial B_m(\vec{r})}{\partial r_e} \stackrel{\text{Eq.(8), (9), (9)A}}{=} \\ &\quad \cdot = \int d^3 r' \epsilon_{\text{stm}} \epsilon_{\text{tie}} m_i \frac{\partial B_m(\vec{r})}{\partial r_e}, \quad \vec{m} \text{ is the dipole moment of the distribution, } \vec{j}(\vec{r}') \\ &= (\delta_{se} \delta_{mi} - \delta_{si} \delta_{me}) m_i \frac{\partial B_m}{\partial r_e} = \\ &= m_i \frac{\partial B_i}{\partial r_s} - m_s \frac{\partial B_m}{\partial r_m} \Rightarrow \frac{\partial B_m}{\partial r_m} = \vec{\nabla} \cdot \vec{B} = 0\end{aligned}$$

$$\Rightarrow \boxed{\vec{F} = m_k \vec{\nabla} B_k} \quad \vec{F} = 0 \text{ unless } \vec{B} \neq \text{const}$$

if  $\vec{m}$  is fixed,  $\vec{F} = \vec{\nabla}(\vec{m} \cdot \vec{B})$ . In this case  $\vec{F} = -\vec{\nabla} \cdot \hat{V}_B(\vec{r})$ , the potential energy  $\hat{V}_B = -\vec{m} \cdot \vec{B}(\vec{r})$

Alternative expression for  $\vec{F}$  is obtained using  
 $\vec{\nabla} \times \vec{B}(F) = 0$  [Recall that  $\vec{B}$  is created by currents different (actually quite far from) the distribution  $\vec{j}(F')$ ]

$$F_s = m_k \frac{\partial}{\partial r_s} B_k = m_k \frac{\partial B_s}{\partial r_k}, \text{ since } \frac{\partial B_k}{\partial r_s} = \frac{\partial B_s}{\partial r_k}, \vec{\nabla} \times \vec{B} = 0$$

$$\Rightarrow F_s = (\vec{m} \cdot \vec{\nabla}) B_s, \quad \vec{F} = (\vec{m} \cdot \vec{\nabla}) \vec{B}$$

Example 12.3 Show that the force on a point magnetic dipole is  $\vec{F} = \vec{\nabla}(\vec{m} \cdot \vec{B})$

$$\square \quad \vec{J}_D(F) = -\vec{m} \times \vec{\nabla} \delta(F - \vec{r}_0)$$

$$\begin{aligned} \vec{F} &= \int d^3r \vec{J}_D \times \vec{B} = \int d^3r \vec{B} \times [\vec{m} \times \vec{\nabla} \delta(F - \vec{r}_0)] \\ &= \int d^3r [\vec{m} \cdot (\vec{B} \cdot \vec{\nabla} \delta(F - \vec{r}_0)) - (\vec{m} \cdot \vec{B}) \vec{\nabla} \delta(F - \vec{r}_0)] \\ &= \int d^3r \vec{m} (-\vec{\nabla} \cdot \vec{B}) \delta(F - \vec{r}_0) + \int d^3r \delta(F - \vec{r}_0) \vec{\nabla}(\vec{m} \cdot \vec{B}) \end{aligned}$$

→ this transition is done by the integration by parts, and using that  $\vec{m}$  is  $\vec{r}$ -independent vector. Now the first term drops since  $\vec{\nabla} \cdot \vec{B} = 0$  and the second term gives

$$\vec{F} = \vec{\nabla}_{r_0} (\vec{m} \cdot \vec{B}(r_0)) \quad \text{P.D.}$$

#### 12.4.2. Dipolar potential energy

$$\hat{V}_B(F) = -\vec{m} \cdot \vec{B}(F)$$

→ true for a fixed dipole moment  $\vec{m}$

## 12.4.4. The dipole Torque

$$\vec{N} = \int d^3 r' \vec{r}' \times \left[ \vec{j}(\vec{r}') \times [\vec{B}(\vec{r}) + (\vec{r}' - \vec{r}) \cdot \vec{\nabla}] \vec{B}(\vec{r}) + \dots \right]$$

$$\approx \int d^3 r' \vec{r}' \times [\vec{j}(\vec{r}') \times \vec{B}(\vec{r})] = \int d^3 r' \vec{j}(\vec{r}') (\vec{F}' \cdot \vec{B}(\vec{r}))$$

$$- \vec{B}(\vec{r}) \int d^3 r' \vec{F}' \cdot \vec{j}(\vec{r}')$$

From Eq.(9)A the second term vanishes

$$\vec{N} = \int d^3 r' \vec{j}(\vec{r}') (\vec{F}' \cdot \vec{B}(\vec{r}'))$$

$$\Rightarrow N_e = \int d^3 r' j_e(\vec{r}') r'_k B_k(\vec{r}) = \text{Elike } m_i B_k(\vec{r}) =$$

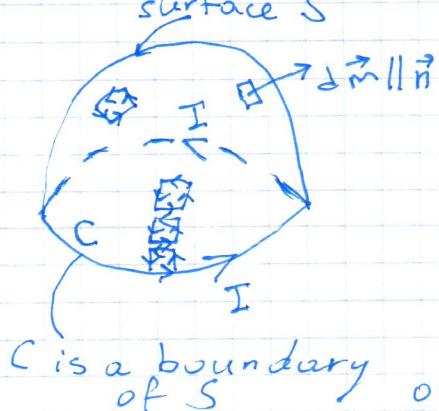
$$= [\vec{m} \times \vec{B}(\vec{r})]_e$$

$$\Rightarrow \boxed{\vec{N} = \vec{m} \times \vec{B}(\vec{r})}$$

## 11.3 Magnetic Dipole Layers: Ampère's Theorem

If  $S$  is an open surface and dipoles are oriented normal to the surface, the magnetic field produced by  $S$  is identical to the magnetic field produced by a current flowing around the boundary of  $S$

Assumption: Each surface element  $d\vec{s}\hat{n}$  (of surface  $S$ ) area  $ds$  and  $\hat{n}$  being a unit normal vector, perpendicular to the surface) carries a dipole moment  $d\vec{m} = I ds \hat{n}$ , where  $I = \frac{d\vec{m}}{ds}$  is constant all over the surface  $S$



$$\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi} \int_S d\vec{m} \times \frac{\vec{r} - \vec{r}_S}{|\vec{r} - \vec{r}_S|^3} = \frac{\mu_0 I}{4\pi} \int_S dS \hat{n} \times \vec{P}_S \left( \frac{1}{|\vec{r} - \vec{r}_S|} \right),$$

where  $\vec{r}_S \in S$  is running integration variable that runs over the surface  $S$ ,  $\vec{r}$  is the observation point,  $d\vec{m} = IdS \hat{n}$  is the dipole moment of a surface element  $dS$ , and in  $\vec{P}_S$  the special derivatives act on  $\vec{r}_S$  and not on  $\vec{r}$ .

Take arbitrary, constant vector  $\vec{z}$ ,

$$\begin{aligned} \vec{z} \cdot \vec{A}(\vec{r}) &= \frac{\mu_0 I}{4\pi} \int_S dS \vec{z} \cdot \left[ \hat{n} \times \vec{P}_S \left( \frac{1}{|\vec{r} - \vec{r}_S|} \right) \right] = \\ &= \frac{\mu_0 I}{4\pi} \int_S dS \hat{n} \cdot \left[ \vec{P}_S \left( \frac{1}{|\vec{r} - \vec{r}_S|} \right) \times \vec{z} \right] \end{aligned}$$

Using again (Appendix)  $\vec{\nabla} \times [\vec{C} f(\vec{r})] = -\vec{C} \times \vec{\nabla} f(\vec{r})$ ,

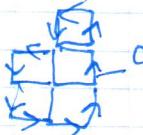
$$\vec{z} \cdot \vec{A}(\vec{r}) = \frac{\mu_0 I}{4\pi} \int_S dS \cdot \left[ \vec{P}_S \times \frac{\vec{z}}{|\vec{r} - \vec{r}_S|} \right] = \frac{\mu_0}{4\pi} \oint_C d\vec{l} \cdot \frac{\vec{z}}{|\vec{r} - \vec{r}_S|}$$

$\uparrow$  Stokes

As  $\vec{z}$  is arbitrary,

$\vec{A}(\vec{r}) = \frac{\mu_0 I}{4\pi} \oint_C \frac{d\vec{l}}{|\vec{r} - \vec{r}_S|}$ , which is vector potential of a filamentary current encircling  $S$   $\blacksquare$

Simple meaning:  $d\vec{m}$  is from current  $I$  going around a small square,  $dS$ ,  $\cancel{\downarrow} I$ . for the two

 current such squares the adjacent segments cancel  $\Rightarrow$  the boundary remains (Fig.)