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## Ch.5.1 Magnetostatics: Introduction and Definitions

J.O.J. No magnetic charges in nature  $\Rightarrow$

$\Rightarrow$  the basic quantity is magnetic dipole

that react on a magnetic field according to  
the law

$$\vec{N} = \vec{\mu} \times \vec{B}, \text{ where}$$

$\vec{N}$  is the torque acting on a dipole and trying to rotate it;  $\vec{\mu}$  is the magnetic moment of the dipole, and  $\vec{B}$  is a) magnetic flux density  
b) magnetic induction. I prefer to call it  $\vec{B}$ -field (to distinguish it from the  $\vec{H}$ -field in a most obvious way).

The relation  $\vec{N} = \vec{\mu} \times \vec{B}$  is that what alignes the compass needle along the field.

The real progress in understanding magnetism started with experimental discovery of the relation between the magnetic field and currents.

Let's remind ourselves how one describes the electrical current, i.e. a steady flow of electrical charges.

$\vec{J}$  = the current density is the (positive) charge crossing a unit area perpendicular to the flow direction per unit time.

For positive charges  $\vec{J}$  is along the flow

For negative charges  $\vec{J}$  is against the flow,

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because positive charge crossing from left to right is equivalent to negative charge crossing in the opposite direction.

Experimentally electrical charge is conserved:  
for any volume  $V$  bounded by a surface  $S$

$\frac{dQ}{dt} = - \oint_S d\vec{a} \cdot \vec{J}$ , where  $Q$  is the total charge inside the volume  $V$ , and  $\hat{n}$  is the unit normal to the surface  $S$ .



$$\frac{d}{dt} \int_V d^3x \rho + \oint_S d\vec{a} \cdot \vec{J} = 0$$

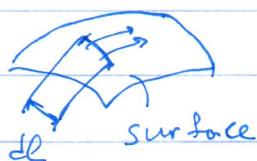
By divergence theorem

$$\int_V d^3x \frac{\partial \rho}{\partial t} + \int_V d^3x \nabla \cdot \vec{J} = 0 \Rightarrow \text{for any } \int_V d^3x \left( \frac{\partial \rho}{\partial t} + \nabla \cdot \vec{J} \right) = 0$$

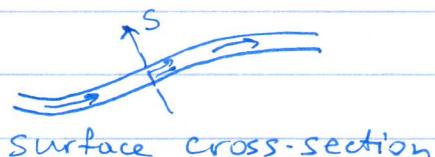
$$\Rightarrow \frac{\partial \rho}{\partial t} + \nabla \cdot \vec{J} = 0 \quad \text{continuity equation.}$$

$$\text{In magneto} \underline{\text{statics}} \quad \frac{\partial \rho}{\partial t} = 0 \Rightarrow \nabla \cdot \vec{J} = 0$$

It is sometimes usefull to introduce a surface current density and just a current.



$\vec{K}$  is the charge crossing a unit length element, of a surface per unit time in direction perpendicular to the element

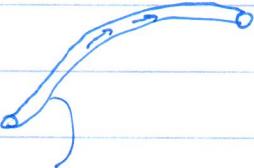


$$\vec{K} = \int d\vec{s} \vec{J}$$

line integral along the direction perpendicular to the surface

Similarly for the filamentary current

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current carrying wire

$$\vec{I} = \int d^2S \vec{J}$$

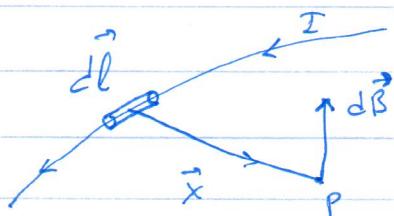
↑ two dimensional integral  
over the perpendicular cross-section  
of a wire.

By continuity  $|\vec{I}| = I$  is constant along the wire.

### Ch. 5.2 Biot-Savart Law

(1819) Oersted: wires carrying electrical current deflect magnetic moments  $\Rightarrow$  currents produce magnetic field

(1820) Biot and Savart, (1820-1825) Ampere : Laws relating magnetic field to currents and Laws of forces between the currents



$$d\vec{B} = \kappa I \frac{d\vec{l} \times \vec{x}}{|x|^3} \quad \text{Biot-Savart}$$

$d\vec{l}$  is the element of length pointing along the current

$k$  depends on units :  $\left\{ \begin{array}{l} k = \frac{\mu_0}{4\pi} \quad \text{SI} \\ k = 1/c \quad \text{CGS} \end{array} \right.$

The meaning of B-S law is that  $\vec{B} = \int d\vec{B}$ , i.e. the magnetic induction is the sum of elementary contributions from each element  $d\vec{l}$ .

One might try to reinterpret B-S law by saying that the magnetic field is essentially due to currents,

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that are moving charges. In this case this interpretation means that the charge  $q$  moving with the current produces the field

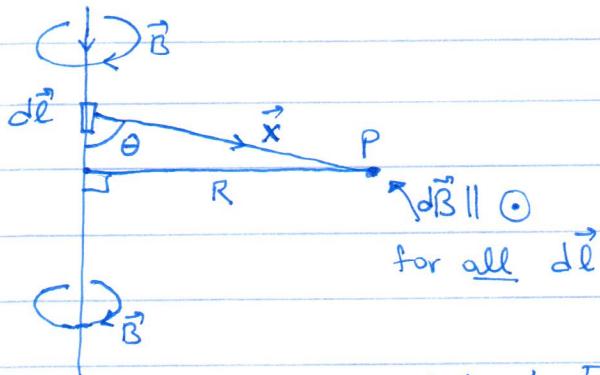
$$\vec{B} = \kappa q \frac{\vec{v} \times \vec{x}}{x^3}$$

we will obtain the field of moving charge at a constant speed, and will find that this expression is only valid in non-relativistic limit  $v \ll c$

Now if the wire is curved, the charges accelerate and we will see that such charge radiates.

Because of these two reasons the only meaningful interpretation of Biot-Savart law is that the total  $\vec{B}$ -field is the sum of all the elementary contributions  $d\vec{B}$  over a closed current loop. The loop may also be closed at infinity.

Example The  $\vec{B}$ -field due to straight infinite wire



$$|\vec{B}| = \frac{\mu_0 I}{4\pi} \int_{-\infty}^{+\infty} \frac{dl}{x^2} \sin \theta$$

$$x = \sqrt{R^2 + l^2}$$

$$\sin \theta = \frac{R}{x} = \frac{R}{\sqrt{R^2 + l^2}}$$

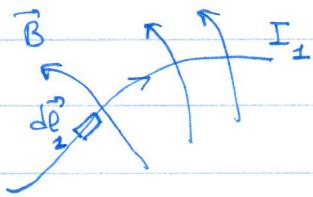
$$\Rightarrow |\vec{B}| = \frac{\mu_0 I R}{4\pi} \int_{-\infty}^{+\infty} \frac{dl}{(R^2 + l^2)^{3/2}} = |\vec{z}| = l/R$$

$$= \frac{\mu_0 I}{4\pi R} \underbrace{\int_{-\infty}^{+\infty} \frac{dz}{(1+z^2)^{3/2}}}_{2} \Rightarrow$$

$$|\vec{B}| = \boxed{\frac{\mu_0 I}{2\pi R}}$$

## Ampere's Law of force

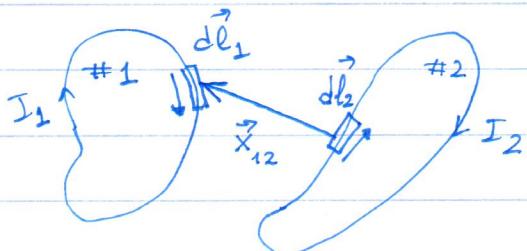
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The force acting on a current element  $I_1 d\vec{l}_1$  in the presence of magnetic induction  $\vec{B}$  is

$$d\vec{F} = I_1 (d\vec{l}_1 \times \vec{B})$$

Consider the case of  $\vec{B}$  produced by the loop #2 and evaluate the total force  $\vec{F}_{1 \leftarrow 2}$  that acts on a loop with the current  $I_1$



The field due to the loop #2 according to Biot-Savart law

$$\vec{B}(\vec{x}_1) = \frac{\mu_0}{4\pi} \oint_{L_2} I_2 \frac{d\vec{l}_2 \times \vec{x}_{12}}{|\vec{x}_{12}|^3}$$

$\int$  integral along loop #2

From Ampere law then

$$\vec{F}_{1 \leftarrow 2} = \oint_{L_1} d\vec{F} = \frac{\mu_0}{4\pi} I_1 I_2 \oint_{L_1} \oint_{L_2} \frac{d\vec{l}_1 \times (d\vec{l}_2 \times \vec{x}_{12})}{|\vec{x}_{12}|^3}$$

To see that the Newton 3-rd law is satisfied,

i.e.  $\vec{F}_{1 \leftarrow 2} = -\vec{F}_{2 \leftarrow 1}$

$$\frac{d\vec{l}_1 \times (d\vec{l}_2 \times \vec{x}_{12})}{|\vec{x}_{12}|^3} = -(\vec{d}\vec{l}_1 \cdot \vec{d}\vec{l}_2) \frac{\vec{x}_{12}}{|\vec{x}_{12}|^3} + \vec{d}\vec{l}_2 \frac{(\vec{d}\vec{l}_1 \cdot \vec{x}_{12})}{|\vec{x}_{12}|^3}$$

Recall that  $\frac{\vec{x}_{12}}{|\vec{x}_{12}|^3} = -\vec{\nabla}_{x_1} \frac{1}{|x_{12}|} \Rightarrow$

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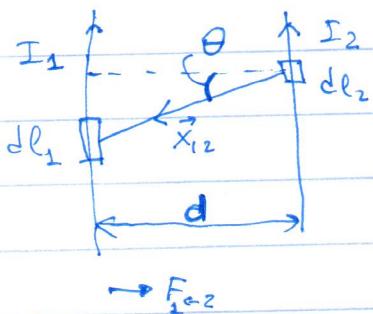
$$\oint d\vec{l}_1 \cdot \frac{\vec{x}_{12}}{|\vec{x}_{12}|^3} = - \oint d\vec{l}_1 \cdot \hat{\vec{x}}_{x_2} \frac{1}{|\vec{x}_{12}|} = 0 \quad \text{provided the two loops do not cross.}$$

Assuming they do not we are left with the first term:

$$\vec{F}_{1 \leftarrow 2} = - \frac{\mu_0}{4\pi} I_1 I_2 \oint_{l_1 l_2} \frac{(d\vec{l}_1 \cdot d\vec{l}_2) \vec{x}_{12}}{|\vec{x}_{12}|^3}$$

$$\text{It's clear now that } \vec{F}_{1 \leftarrow 2} = - \vec{F}_{2 \leftarrow 1} \quad (\vec{x}_{12} = -\vec{x}_{21})$$

Consider two parallel and infinitely long wires:



$$\begin{aligned} \frac{d|\vec{F}_{1 \leftarrow 2}|}{dl} &= \frac{\mu_0 I_1 I_2}{4\pi} \int_{-\infty}^{+\infty} dl_2 \frac{|\vec{x}_{12}| \cos \theta}{|\vec{x}_{12}|^3} = \\ &= \frac{\mu_0 I_1 I_2}{4\pi} \int_{-\infty}^{+\infty} dl \frac{d}{(e^2 + d^2)^{3/2}} \Rightarrow \end{aligned}$$

$$\frac{d|\vec{F}_{1 \leftarrow 2}|}{dl} = \frac{\mu_0 I_1 I_2}{2\pi d}$$

parallel currents  $\Rightarrow$  attraction

anti-parallel currents  $\Rightarrow$  repulsion.

For generic current density  $\vec{J}(\vec{r})$  in the presence of a magnetic field  $\vec{B}(F)$ , the force acting on a current distribution  $\vec{J}(\vec{r})$  is

$$\vec{F} = \int d^3x \vec{J}(\vec{x}) \times \vec{B}(\vec{x})$$

Similarly the torque

$$\vec{N} = \int d^3x \vec{x} \times [\vec{J}(\vec{x}) \times \vec{B}(\vec{x})]$$

## Ch. 5.3 Differential Equations of Magnetostatics and

J.D.J. Ampere's Law

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The basic law of magnetostatics

$$\vec{B}(\vec{x}) = \frac{\mu_0}{4\pi} \int d^3x' \vec{J}(\vec{x}') \times \frac{\vec{x} - \vec{x}'}{|\vec{x} - \vec{x}'|^3} \quad (1)$$

is very similar to  $\vec{E}(\vec{x}) = \frac{1}{4\pi\epsilon_0} \int d^3x' \rho(\vec{x}') \frac{\vec{x} - \vec{x}'}{|\vec{x} - \vec{x}'|^3}$

Similar to electostatics it is advantageous to formulate the basic law in differential form

We rewrite (1) using two observations

$$A \quad \frac{\vec{x} - \vec{x}'}{|\vec{x} - \vec{x}'|^3} = -\vec{\nabla}_{\vec{x}} \frac{1}{|\vec{x} - \vec{x}'|}$$

B : For a constant vector  $\vec{C}$  and a scalar function  $f(\vec{x})$

$$\vec{\nabla} \times [\vec{C} f(\vec{x})] = -\vec{C} \times \vec{\nabla} f(\vec{x})$$

$$D \quad \left\{ \vec{\nabla} \times [\vec{C} f(\vec{x})] \right\}_k = \epsilon_{kem} \frac{\partial}{\partial x_e} C_m f(\vec{x}) = -\epsilon_{kme} C_m \frac{\partial f(\vec{x})}{\partial x_e} = \\ = -[\vec{C} \times \vec{\nabla} f(\vec{x})]_k \quad \blacksquare$$

A, B  $\Rightarrow$  Eq. (1) gives

$$\vec{B}(\vec{x}) = + \frac{\mu_0}{4\pi} \int d^3x' \vec{J}(\vec{x}') \times \left( -\vec{\nabla}_{\vec{x}} \frac{1}{|\vec{x} - \vec{x}'|} \right) = \frac{\mu_0}{4\pi} \vec{\nabla}_{\vec{x}} \times \int d^3x' \frac{\vec{J}(\vec{x}')}{|\vec{x} - \vec{x}'|} \quad (2)$$

From Eq.(2)  $\Rightarrow$   $\boxed{\vec{\nabla} \cdot \vec{B} = 0}$

To fix the field (up to a B.C.) we need a curl

(3)

Take curl of Eq. (2)

$$\bar{\nabla} \times \bar{B}(\vec{x}) = \frac{\mu_0}{4\pi} \bar{\nabla}_x \times \bar{\nabla}_x \times \int d^3x' \frac{\bar{J}(\vec{x}')}{|\vec{x} - \vec{x}'|}$$

Recall: for a field  $\vec{F}(\vec{x})$  we have

$$\bar{\nabla} \times (\bar{\nabla} \times \vec{F}) = \bar{\nabla}(\bar{\nabla} \cdot \vec{F}) - \bar{\nabla}^2 \vec{F}$$

$$\square [\bar{\nabla} \times (\bar{\nabla} \times \bar{F})]_k = \epsilon_{kem} \frac{\partial}{\partial x_e} \epsilon_{mst} \frac{\partial}{\partial x_s} F_t = \\ = (\delta_{ks} \delta_{et} - \delta_{kt} \delta_{es}) \frac{\partial}{\partial x_e} \frac{\partial}{\partial x_s} F_t = \frac{\partial}{\partial x_k} \left( \frac{\partial F_e}{\partial x_e} \right) - \sum_s \frac{\partial^2}{\partial x_s^2} F_k \quad \boxed{\text{OK}}$$

$$\bar{\nabla} \times \bar{B}(\vec{x}) = \frac{\mu_0}{4\pi} \bar{\nabla}_x \int d^3x' \bar{J}(\vec{x}') \cdot \bar{\nabla}_x \left( \frac{1}{|\vec{x} - \vec{x}'|} \right) - \frac{\mu_0}{4\pi} \int d^3x' \bar{J}(\vec{x}') \bar{\nabla}_x^2 \left( \frac{1}{|\vec{x} - \vec{x}'|} \right) \quad (4)$$

The first term of Eq. (4):

$$\int d^3x' \bar{J}(\vec{x}') \cdot \bar{\nabla}_x \left( \frac{1}{|\vec{x} - \vec{x}'|} \right) = - \int d^3x' \bar{J}(\vec{x}') \cdot \bar{\nabla}_{x'} \left( \frac{1}{|\vec{x} - \vec{x}'|} \right) \stackrel{\text{By parts}}{=} \\ = \int d^3x' (\bar{\nabla}_{x'} \cdot \bar{J}(\vec{x}')) \left( \frac{1}{|\vec{x} - \vec{x}'|} \right) = 0 \quad \text{since} \quad \bar{\nabla} \cdot \bar{J} = 0$$

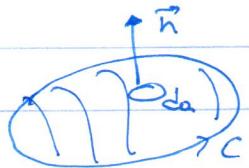
In the second term of (5) apply  $\bar{\nabla}_x^2 \frac{1}{|\vec{x} - \vec{x}'|} = -4\pi \delta(\vec{x} - \vec{x}')$

$$\bar{\nabla} \times \bar{B}(\vec{x}) = - \frac{\mu_0}{4\pi} \int d^3x' \bar{J}(\vec{x}') (-4\pi) \delta(\vec{x} - \vec{x}') = \mu_0 \bar{J}(\vec{x})$$

$$\Rightarrow \text{Ampere Law} \quad \boxed{\bar{\nabla} \times \bar{B} = \mu_0 \bar{J}} \quad (5)$$

To rewrite it in integral form

(P.9)



$$\int_S d\vec{a} \cdot (\vec{\nabla} \times \vec{B}) = \mu_0 \int_S d\vec{a} \cdot \vec{n} \cdot \vec{J}$$

$$\text{Stokes} \Rightarrow \boxed{\oint_C d\vec{l} \cdot \vec{B} = \mu_0 I} \quad (6)$$

Ampere's law in integral form.

## Ch. 5.4 Vector Potential

J.D.J.

$$\vec{\nabla} \times \vec{B} = \mu_0 \vec{J}, \quad \vec{\nabla} \cdot \vec{B} = 0 \quad (7)$$

$$\vec{\nabla} \cdot \vec{B} = 0 \Rightarrow \vec{B} = \vec{\nabla} \times \vec{A}$$

(8)

We indeed saw, Eq.(2) p.7 that

$$\vec{A} = \frac{\mu_0}{4\pi} \int d^3x' \frac{\vec{J}(\vec{x}')}{|\vec{x} - \vec{x}'|} + \vec{\nabla} \psi, \quad (9)$$

Where  $\psi$  is arbitrary. This addition shows that if we have some  $\vec{A}$  such that  $\vec{B} = \vec{\nabla} \times \vec{A}$  we can use  $\vec{A} + \vec{\nabla} \psi$  instead since  $\vec{\nabla} \times (\vec{A} + \vec{\nabla} \psi) = \vec{\nabla} \times \vec{A} = \vec{B}$  this arbitrariness is a gauge freedom and transformation

$\vec{A} \rightarrow \vec{A} + \vec{\nabla} \psi$  a gauge transformation.

$$\text{Eq.8} \rightarrow \text{Eq.7} \Rightarrow \vec{\nabla} \times (\vec{\nabla} \times \vec{A}) = \mu_0 \vec{J} \Rightarrow$$

$$\Rightarrow \vec{\nabla}(\vec{\nabla} \cdot \vec{A}) - \vec{\nabla}^2 \vec{A} = \mu_0 \vec{J}$$

In a Coulomb gauge  $\vec{\nabla} \cdot \vec{A} = 0 \Rightarrow$  Each component of a vector potential satisfies Poisson equation

$$\nabla^2 \vec{A} = -\mu_0 \vec{J}$$

From electrostatics, in unbounded space the solutions are

$$\vec{A}(\vec{x}) = \frac{\mu_0}{4\pi} \int d^3x' \frac{\vec{J}(x')}{|\vec{x} - \vec{x}'|} \quad (11)$$

Let's compare Eq.(9) and Eq.(11): it follows that the choice  $\vec{\nabla} \cdot \vec{A} = 0$  is equivalent to setting  $\psi = \text{const}$  in Eq.(9).

Could we foresee it? Yes:

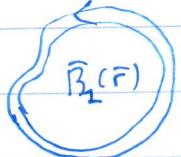
$$\text{Recall that as } \vec{\nabla} \cdot \vec{J} = 0, \quad \vec{\nabla}_x \int d^3x' \frac{\vec{J}(x')}{|\vec{x} - \vec{x}'|} = 0$$

$$\Rightarrow \vec{\nabla} \cdot \vec{A} = \vec{\nabla}^2 \psi$$

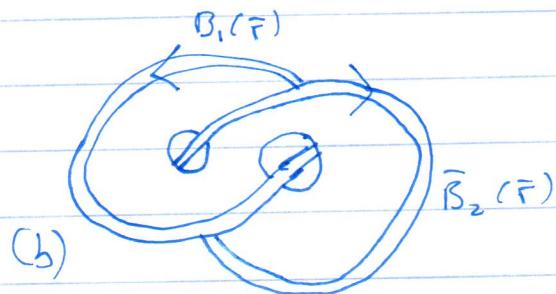
$\Rightarrow \vec{\nabla} \cdot \vec{A} = 0$  means  $\vec{\nabla}^2 \psi = 0$ , which has  $\psi = \text{const}$  as the only well behaved solution in an unbounded space.

Example 10.7 A.Z. The helicity  $h = \int d^3r \vec{A} \cdot \vec{B}$  is a quantitative measure of a topological complexity of a magnetic field configuration. It is used extensively in solar physics and other situations. To illustrate the idea, define a flux tube to be a bundle of parallel magnetic field lines where the same magnetic flux passes through every cross-section of the tube. Figure shows two flux tubes which close on themselves. The two tubes linked on Fig(a) and unlinked on Fig.(b).

- (a) Show that  $h=0$  when the tubes are unlinked  
 and  $h = 2\Phi_1\Phi_2$  when the tubes are linked  
 (b) Find the conditions required to make the  
 definition of  $h$  gauge-invariant



(a)



(b)

Solution  $\vec{A} = \vec{A}_1 + \vec{A}_2$ ,  $\vec{B}_{s,2} = \vec{\nabla} \times \vec{A}_{s,2}$  is  
 confined to volume  $V_{1,2}$  respectively.

$$h = \int_{V_1} d^3r \vec{A} \cdot \vec{B}_1 + \int_{V_2} d^3r \vec{A} \cdot \vec{B}_2$$

in each tube  $d^3r = dl ds$ ,  $\vec{B} \parallel dl \parallel ds$

↗ cross-sectional  
 ↓ area  
 ↗ elementary length

$$\int_{V_1} d^3r \vec{A} \cdot \vec{B}_1 = \oint_{C_1} dl \int_{S_1} ds \vec{A} \cdot \vec{B}_1 = \oint_{C_1} dl \int_{S_1} ds (\vec{A}_{\parallel} + \vec{A}_{\perp}) \cdot \vec{B}_1$$

where  $\vec{A}_{\parallel}$ ,  $\vec{A}_{\perp}$  are parallel and perpendicular to  
 the tube  $dl$  components of  $\vec{A}$

$\vec{B}_1$  is parallel to  $dl$  by definition  $\Rightarrow$

$$= \oint_{C_1} dl \int_{S_1} ds A_{\parallel} \cdot \vec{B}_1 = \oint_{C_1} dl \int_{S_1} ds (A_{1,\parallel} + A_{2,\parallel}) \cdot \vec{B}_1 =$$

$$= \oint_{C_1} dl \int_{S_1} ds A_{1,\parallel} \cdot \vec{B}_1 + \oint_{C_1} dl \int_{S_1} ds A_{2,\parallel} \cdot \vec{B}_1$$

The second term (the tubes are narrow)

$$\oint_{C_1} d\ell, \int dS_1 A_{2,||} B_1 = \oint_{C_1} d\ell, A_{2,||} \int dS_1 B_1 = \oint_{C_1} d\ell, A_{2,||} \Phi_1$$

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$$= \Phi_2 \Phi_1 \text{ if loops are linked}$$

$$0 \text{ if loops are unlinked}$$

$$\text{The first term } \oint_{C_1} d\ell, A_{2,||} \Phi_1 = \Phi_1 \oint_{C_1} d\ell, A_{2,||} = 0$$

since



$$\oint_{C_1} d\ell, A_{2,||} = \int d\vec{S} \cdot \vec{B}_1 = 0$$

since  $d\vec{S} \perp (d\ell_1 \parallel A_{2,||})$  on a tube

(b)

$$h' = \int d^3r (\bar{A} + \bar{\nabla} \times \bar{A}) \cdot \bar{B} = \int d^3r \bar{A} \cdot \bar{B} + \int d^3r \bar{\nabla} \times \bar{A} \cdot \bar{B} =$$

$$= h + \int d^3r \bar{\nabla} \times \bar{A} \cdot \bar{B} = h + \int d^3r \bar{\nabla} \cdot (\bar{A} \times \bar{B}) =$$

$\uparrow$   
 $\bar{\nabla} \cdot \bar{B} = 0$

$$= h + \oint_S da \bar{n} \cdot \bar{A} \times \bar{B} \Rightarrow h' = h \text{ if } \bar{n} \cdot \bar{B} = 0$$

on a surface that bounds  
the field.

it is well justified for very large volume  
with localized current distribution (sun).