

# Ch. 3.11 Expansion of Green Functions in Cylindrical Coordinates (J.D.J.)

p. ①

## Ch. 8.5.4 (A.Z)

We consider a Green function in empty space,  $|{\vec{x}} - {\vec{x}'}|^{-1}$  in terms of Cylindrical Coordinates. The procedure is such that it can be easily generalized to regions bounded by cylindrically symmetric surfaces.

The method we adopt is often referred to as The METHOD OF Direct Integration.

Start with the defining equation in cylindrical coordinates.

$$\nabla_{\vec{x}}^2 G({\vec{x}}, {\vec{x}'}) = - \frac{4\pi}{\rho} \delta(\rho - \rho') \delta(\phi - \phi') \delta(z - z') \quad [= -4\pi \delta({\vec{x}} - {\vec{x}'}))] \quad ①$$

Represent 2 out of 3  $\delta$ -functions as follows

$$\delta(z - z') = \frac{1}{2\pi} \int_{-\infty}^{+\infty} dk e^{ik(z-z')} = \frac{1}{\pi} \int_0^{\infty} dk \cos[k(z-z')] \quad ②$$

$$\delta(\phi - \phi') = \frac{1}{2\pi} \sum_{m=-\infty}^{+\infty} e^{im(\phi-\phi')} \quad ③$$

It is natural to expand the Green function in a similar way:

$$G({\vec{x}}, {\vec{x}'}) = \frac{1}{2\pi^2} \int_0^{\infty} dk \sum_{m=-\infty}^{+\infty} e^{im(\phi-\phi')} \cos[k(z-z')] g_m(k, \rho, \rho') \quad ④$$

Recall

$$\nabla^2 \psi = \frac{1}{\rho} \frac{\partial}{\partial \rho} \left( \rho \frac{\partial \psi}{\partial \rho} \right) + \frac{1}{\rho^2} \frac{\partial^2 \psi}{\partial \phi^2} + \frac{\partial^2 \psi}{\partial z^2} \quad ⑤$$

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Eq. (4)  $\rightarrow$  RHS Eq. (1) using (5) { }  $\Rightarrow$   
 Eq. (2), (3)  $\rightarrow$  LHS Eq. (1)

$$\Rightarrow \frac{1}{2\pi^2} \sum_{m=-\infty}^{+\infty} \int_0^\infty dk e^{im(\phi-\phi')} \cos [K(z-z')] \times$$

$$\times \left\{ \frac{1}{\rho} \frac{\partial}{\partial \rho} \left( \rho \frac{\partial g_m}{\partial \rho} \right) - \left( K^2 + \frac{m^2}{\rho^2} \right) g_m \right\} =$$

$$= \frac{1}{2\pi^2} \sum_{n=-\infty}^{+\infty} \int_0^\infty dk e^{im(\phi-\phi')} \cos [K(z-z')] \times$$

$$\times \left\{ -\frac{4\pi}{\rho} \delta(\rho - \rho') \right\}$$

The only way to satisfy (6) is to require

$$\frac{1}{\rho} \frac{d}{d\rho} \left( \rho \frac{dg_m}{d\rho} \right) - \left( K^2 + \frac{m^2}{\rho^2} \right) g_m = -\frac{4\pi}{\rho} \delta(\rho - \rho')$$

Consider the equation defining the modified Bessel functions

$$\frac{d^2 R}{dx^2} + \frac{1}{x} \frac{dR}{dx} - \left( 1 + \frac{\nu^2}{x^2} \right) R = 0 \quad (8)$$

The solutions of (8) are called modified Bessel functions that are just a regular Bessel functions of purely imaginary argument. The usual choice of the two linearly independent solutions of (8) are

$$I_\nu(x) = i^{-\nu} J_\nu(ix), \quad K_\nu(x) = \frac{\pi}{2} i^{\nu+1} H_\nu^{(1)}(ix)$$

Recall  $H_\nu^{(1)}(x) = J_\nu(x) - i N_\nu(x)$  are Hankel,  $N_\nu$  is Neumann functions

$$H_\nu^{(2)}(x) = J_\nu(x) - i N_\nu(x)$$

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$$I_m(x) K_m'(x) - I_m'(x) K_m(x) = -\frac{1}{x}$$

$$\Rightarrow A = 4\pi \Rightarrow g_m(k_p, p') = 4\pi I_m(k_p) K_m(k_p)$$

$$\text{Eq. (4)} \quad G(\vec{x}, \vec{x}') = \frac{1}{|\vec{x} - \vec{x}'|} =$$

$$= \frac{2}{\pi} \sum_{m=-\infty}^{+\infty} \int dk e^{im(\phi-\phi')} \cos[k(z-z')] I_m(k_p) K_m(k_p) \quad (10)$$

$$= \frac{4}{\pi} \int_0^\infty dk \cos[k(z-z')] \times$$

$$\times \left\{ \frac{1}{2} I_0(k_p) K_0(k_p) + \sum_{m=1}^{\infty} \cos[m(\phi-\phi')] I_m(k_p) K_m(k_p) \right\} \quad (11)$$

$$\text{For } \vec{x}' = 0, \quad |\vec{x} - \vec{x}'|^{-1} = |\vec{x}|^{-1} = (p^2 + z^2)^{-1/2}$$

$$\vec{x}' = 0 \Rightarrow p' = 0 \quad I_m(k_p) \xrightarrow{k_p \ll 1} (k_p)^m \frac{1}{2^m} \frac{1}{\Gamma(m+1)}$$

only  $m=0$  survives the limit  $k_p \rightarrow 0, I_m(0) = \delta_{m,0}$

$$\Rightarrow \boxed{\frac{1}{\sqrt{p^2 + z^2}} = \frac{2}{\pi} \int_0^\infty dk \cos k_z K_0(k_p)} \Rightarrow \text{a useful integral} \quad (12)$$

$$p^2 \rightarrow R^2 = p^2 + p'^2 - 2pp' \cos(\phi - \phi')$$

$$(12) \Rightarrow (p^2 + p'^2 + z^2 - 2pp' \cos(\phi - \phi'))^{-1/2} =$$

$$= \frac{2}{\pi} \int_0^\infty dk \cos k_z K_0[k(p^2 + p'^2 - 2pp' \cos(\phi - \phi'))] \quad (13)$$

$$\text{But } (p^2 + p'^2 + z^2 - 2pp' \cos(\phi - \phi'))^{-1/2} = |\vec{x} - \vec{x}'|^{-1} \text{ for } z' = 0$$

$\Rightarrow$  as a result Eq. (11) gives

We will need the asymptotic properties of these solutions of Eq.(8)

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$$x \ll 1 \quad I_\alpha(x) \rightarrow \frac{1}{\Gamma(\alpha+1)} \left(\frac{x}{2}\right)^\alpha$$

$$K_\alpha(x) \rightarrow \begin{cases} -\left[\ln\left(\frac{x}{2}\right) + \gamma_E + \dots\right], & \alpha=0 \\ \frac{\Gamma(\alpha)}{2} \left(\frac{2}{x}\right)^\alpha & \alpha \neq 0 \end{cases}$$

$$x \gg 1, \alpha \quad I_\alpha(x) \rightarrow \frac{1}{\sqrt{2\pi x}} e^x \left[ 1 + O\left(\frac{1}{x}\right) \right]$$

$$K_\alpha(x) \rightarrow \sqrt{\frac{\pi}{2x}} e^{-x} \left[ 1 + O\left(\frac{1}{x}\right) \right]$$

It follows that  $I_\alpha$  is regular (singular) at origin (infinity) respectively, and  $K_\alpha$  is regular (singular) at infinity (origin) respectively.

These properties together with the comparison between Eqs. (7) and (8) imply that

$$g_m(k, p, p') = \begin{cases} C_1 K_m(kp), & p > p' \\ C_2 I_m(kp), & p < p' \end{cases}$$

where  $C_1, C_2$  are constants that depend on  $k, p'$ .

These can be determined by two conditions:

1) Continuity of  $g_m$  at  $p = p'$

2) The cusp discontinuity at  $p = p'$ : jump in the derivative

The physical meaning of the jump is  $\hat{n} \cdot (\vec{E}_1 - \vec{E}_2) = \frac{\sigma}{\epsilon_0}$  for the case of a point charge at  $p = p'$

The second condition can be obtained by applying

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the  $\lim_{\epsilon \rightarrow 0} \int_{p'-\epsilon}^{p'+\epsilon} dg g [ ]$  operation on Eq.(7). It reads

$$\frac{dg_m}{dp} \Big|_{p=p'+\epsilon} - \frac{dg_m}{dp} \Big|_{p=p'-\epsilon} = -\frac{4\pi}{p'}$$

(9)

We can simplify the algebra by noticing that since  $g_m = g_{-m}$ , Eq.(4) defines function symmetric under  $\phi \leftrightarrow \phi'$  and  $z \leftrightarrow z'$ . The symmetry

$G(\vec{z}, \vec{x}') = G(\vec{x}', \vec{z})$  then implies the symmetry

$g_m(k, p, p') = g_m(k, p', p)$ , We can use this property to look for the solution in the form

$$g_m(k, p, p') = A I_m(k p_<) K_m(k p_>)$$

with yet unknown constant  $A$ .

Note that  $g_m$  defined in this way is continuous:

$$\lim_{\epsilon \rightarrow 0} g_m(k, p'+\epsilon, p') = A I_m(k p') K_m(k(p'+\epsilon)) = A I_m(k p') K_m(k p')$$

$$\lim_{\epsilon \rightarrow 0} g_m(k, p'-\epsilon, p') = A I_m(k(p'-\epsilon)) K_m(k p') = A I_m(k p') K_m(k p')$$

as the two limit agree  $g_m$  is continues function of  $p$ .

It remains to satisfy the condition Eq.9 which becomes

$$A [I_m(x') K'_m(x') - I'_m(x') K_m(x')] = -\frac{4\pi}{x'}$$

where  $K' \equiv \frac{dK}{dx}$ ,  $I' \equiv \frac{dI}{dx}$ , and  $x' \equiv k p'$ ,

It can be shown that the modified Bessel functions normalized in the following way:

$$(p^2 + p'^2 + z^2 - 2pp' \cos(\phi - \phi'))^{-1/2} =$$

$$= \frac{4}{\pi} \int_0^\infty dk \cos kz \left\{ \frac{1}{2} I_0(kp_<) K_0(kp_>) + \sum_{m=1}^{\infty} \cos[m(\phi - \phi')] I_m(kp_<) K_m(kp_>) \right\} \quad (14)$$

(13)  $\equiv$  (14) for all  $z \Rightarrow$

$$K_0(k \sqrt{p^2 + p'^2 - 2pp' \cos(\phi - \phi')}) =$$

$$= I_0(kp_<) K_0(kp_>) + 2 \sum_{m=1}^{\infty} \cos[m(\phi - \phi')] I_m(kp_<) K_m(kp_>) \quad (15)$$

holds for all  $k$ .

We are allowed to take  $k \rightarrow 0$  limit in Eq. (15)

$$K_0(k \sqrt{p^2 + p'^2 - 2pp' \cos(\phi - \phi')}) = -\log \frac{k}{2} - \gamma_E$$

$$- \log \sqrt{p^2 + p'^2 - 2pp' \cos(\phi - \phi')} + O(\frac{1}{k})$$

$$I_0(kp_<) = 1 + O(\frac{1}{k}); K_0(kp_>) = -\log \frac{k}{2} - \log p_> + O(\frac{1}{k}) - \gamma_E$$

$m \neq 0$ :

$$I_m(kp_<) K_m(kp_>) = \frac{1}{\Gamma(m+1)} \left( \frac{kp_<}{2} \right)^m \frac{\Gamma(m)}{2} \left( \frac{2}{kp_>} \right)^m = \frac{1}{2^m} \left( \frac{p_<}{p_>} \right)^m$$

$$\Rightarrow -\log \frac{k}{2} - \gamma_E - \log \sqrt{p^2 + p'^2 - 2pp' \cos(\phi - \phi')} =$$

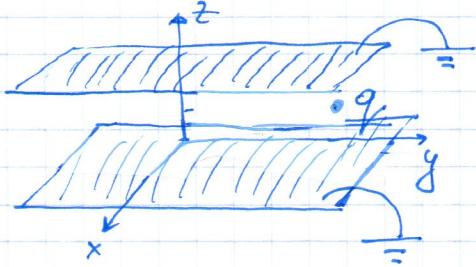
$$= -\log \frac{k}{2} - \gamma_E - \log p_> + 2 \sum_{m=1}^{\infty} \cos[m(\phi - \phi')] \frac{1}{2^m} \left( \frac{p_<}{p_>} \right)^m$$

$$\Rightarrow \log \frac{1}{p^2 + p'^2 - 2pp' \cos(\phi - \phi')} =$$

$$= 2 \log \frac{1}{p_>} + 2 \sum_{m=1}^{\infty} \frac{1}{m} \left( \frac{p_<}{p_>} \right)^m \cos[m(\phi - \phi')]$$

Example 8.4 (A.Z.) Find the potential between two

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infinite grounded plates coincident with  $z=0$  and  $z=d$  when a point charge  $q$  is interposed between them at  $z=z_0$ .

Find the charge induced on the  $z=d$  plate.

Solution: The problem has cylindrical symmetry.

So we follow exactly the same procedure as in the open space with the following change:

$$\delta(z - z') = \frac{1}{\pi} \int_0^\infty dk \cos[k(z - z')] \text{ is replaced by}$$

$$\delta(z - z') = \frac{2}{d} \sum_{n=1}^{\infty} \sin\left[\frac{n\pi z}{d}\right] \sin\left[\frac{n\pi z'}{d}\right]$$

Namely we use another complete set of functions that is consistent with vanishing of the potential on the plates:  $\Phi(z=0) = \Phi(z=d) = 0$ .

We've met this set,  $\{\sin \frac{n\pi z}{d}\}_{n=1}^{\infty}$  when solving the 2D problem in the region  $\{0 < x < L, 0 < y < \infty\}$ , (Ch. 2.10 J.D.I.)

As a result of this replacement we get (see Eq. 10)

$$G_D(\vec{x}, \vec{x}') = \frac{4}{d} \sum_{m=-\infty}^{+\infty} \sum_{n=1}^{\infty} e^{im(\phi - \phi')} \sin\left[\frac{n\pi z}{d}\right] \sin\left[\frac{n\pi z'}{d}\right] \times \\ \times I_m\left(\frac{n\pi \rho_c}{d}\right) K_m\left(\frac{n\pi \rho_s}{d}\right)$$

Now recall that this Green function satisfies

$$\nabla_x^2 G_D(\vec{x}, \vec{x}') = -4\pi \delta(\vec{x} - \vec{x}')$$

Compare it with the Poisson equation

$\nabla^2 \Phi = -\frac{1}{\epsilon_0} \rho$  to deduce that the Green function we found is the potential due to the charge  $\rho(\vec{x}) = 4\pi\epsilon_0 \delta(\vec{x} - \vec{x}')$

It means that the potential due to charge  $q$  localized at  $\vec{x}'$  is  $\Phi_{\vec{x}'}(\vec{x}) = \frac{q}{4\pi\epsilon_0} G_D(\vec{x}, \vec{x}')$

$$\Rightarrow \Phi_{\vec{x}'}(\vec{x}) = \frac{q}{\epsilon_0 \pi d} \sum_{n=1}^{\infty} \sum_{m=-\infty}^{+\infty} e^{im(\phi-\phi')} \sin\left[\frac{n\pi z}{d}\right] \sin\left[\frac{n\pi z'}{d}\right] \times \\ \times I_m\left(\frac{n\pi \rho s}{d}\right) K_m\left(\frac{n\pi \rho s}{d}\right) \quad (14)$$

It is convenient to choose the coordinate system such that the charge  $q$  is located at  $\vec{x}' = (x', y', z') = (0, 0, z_0)$ . Then  $\rho_s = 0$ , and since  $I_m(0) = \delta_{m,0}$  the general expression (14) reduces to [there is no dependence on  $\phi$  by symmetry]:

$$\Phi_{(0,0,z_0)}(\rho, z) = \frac{q}{\epsilon_0 \pi d} \sum_{n=1}^{\infty} \sin\left[\frac{n\pi z}{d}\right] \sin\left[\frac{n\pi z_0}{d}\right] K_0\left(\frac{n\pi \rho}{d}\right)$$

Note: at  $\rho \gg d$  only  $n=1$  term is important since  $K_m(x \gg 1) \approx \sqrt{\pi/2x} e^{-x}$ .

The charge induced on  $z=d$  plate is

$$Q(d) = \int dx dy \sigma(x, y, z=d) = \epsilon_0 \int_0^{2\pi} d\phi \int_0^\infty d\rho \rho \left. \frac{\partial \Phi}{\partial z} \right|_{z=d}$$

$$\left. \frac{d}{dz} \sin\left[\frac{n\pi z}{d}\right] \right|_{z=d} = (-1)^n \frac{n\pi}{d}$$

$$\int_0^{2\pi} d\phi = 2\pi \quad (\text{no } \phi\text{-dependence})$$

$$Q(d) = \frac{q}{\pi d} 2\pi \int_0^\infty dp \sum_{n=1}^{\infty} (-1)^n \frac{\pi n}{d} \sin\left[\frac{n\pi z_0}{d}\right] K_0\left(\frac{n\pi p}{d}\right)$$

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$$\text{New variable } y = \frac{\pi np}{d}$$

$$Q(d) = \frac{2q}{d} \sum_{n=1}^{\infty} (-1)^n \sin\left[\frac{n\pi z_0}{d}\right] \frac{d}{\pi n} \int_0^\infty dy y K_0(y)$$

$$\begin{aligned} &= \frac{2q}{\pi} \sum_{n=1}^{\infty} (-1)^n \frac{1}{n} \sin\left[\frac{n\pi z_0}{d}\right] \underbrace{\int_0^\infty dy y K_0(y)}_{1''} = \\ &= \frac{2q}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \sin\left[\frac{n\pi z_0}{d}\right] = \boxed{-\frac{z_0}{d} q = Q(d)} \end{aligned}$$

This result can be obtained by using the reciprocity relation.

Note: The expansion of  $(\bar{x} - \bar{x}')^{-1}$  in cylindrical coordinates, (10), (11) is not needed in itself, but becomes useful in problems with cylindrical symmetries.