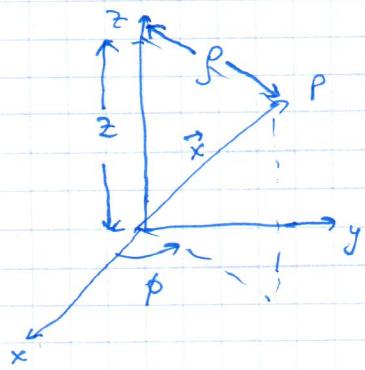


Ch. 3.7 Laplace Equation in Cylindrical Coordinates;

J. D. J. Bessel Functions

P.1



Laplace Equation

$$\frac{\partial^2 \Phi}{\partial \rho^2} + \frac{1}{\rho} \frac{\partial \Phi}{\partial \rho} + \frac{1}{\rho^2} \frac{\partial^2 \Phi}{\partial \phi^2} + \frac{\partial^2 \Phi}{\partial z^2} = 0 \quad (1)$$

Separation of variables

$$\Phi(\rho, \phi, z) = R(\rho) Q(\phi) Z(z) \quad (2)$$

Eq.(2) \Rightarrow Eq.(1) gives

$$R'' Q Z + \frac{1}{\rho} R' Q Z + \frac{1}{\rho^2} R Q'' Z + R Q Z'' = 0$$

Divide by $R Q Z$ to get

$$\frac{R''}{R} + \frac{R'}{\rho R} + \frac{1}{\rho^2} \frac{Q''}{Q} + \frac{Z''}{Z} = 0$$

$$\Rightarrow [Z'' - k^2 Z = 0] \quad (k^2 \text{ can in principle be positive or negative})$$

$$\Rightarrow \text{multiply by } \rho^2 \Rightarrow \rho^2 \frac{R''}{R} + \rho \frac{R'}{R} + \frac{Q''}{Q} + k^2 \rho^2 = 0$$

$$\Rightarrow [Q'' + \sigma^2 Q = 0] \quad (\sigma^2 \text{ also can be positive and negative in general})$$

$$\Rightarrow R'' + \frac{1}{\rho} R' + \left(k^2 - \frac{\sigma^2}{\rho^2}\right) R = 0$$

Take for definiteness $k^2 > 0$

$\sigma^2 > 0$ if the whole range of angles $0 < \phi < 2\pi$ is allowed

$$Z(z) = e^{\pm kz}, \quad Q(\phi) = e^{\pm i\phi}$$

and R satisfies the standard Bessel equation in terms of a dimensionless variable $x = kz$

$$\frac{d^2 R}{dx^2} + \frac{1}{x} \frac{dR}{dx} + \left(1 - \frac{v^2}{x^2}\right) R = 0$$

The solutions are Bessel functions of order v .

The two linearly independent solutions are

$$J_v(x) = \left(\frac{x}{2}\right)^v \sum_{j=0}^{\infty} \frac{(-1)^j}{j! \Gamma(j+v+1)} \left(\frac{x}{2}\right)^{2j}$$

$$J_{-v}(x) = \left(\frac{x}{2}\right)^{-v} \sum_{j=0}^{\infty} \frac{(-1)^j}{j! \Gamma(j-v+1)} \left(\frac{x}{2}\right)^{2j}$$

for v is not an integer

The series representation for $J_{-v}(x)$ does not hold for integer m . In this practically important case the procedure is to use $v \rightarrow m$ limit of the Neumann function (the Bessel function of the second kind):

$$N_v(x) = \frac{J_v(x) \cos v\pi - J_{-v}(x)}{\sin v\pi}$$

$N_v(x)$ is linearly independent of $J_v(x)$ even in the limit $v \rightarrow m$, namely when v is integer.

In many cases the Bessel functions of the third kind, or Hankel functions are more useful :

$$H_v^{(1)}(x) = J_v(x) + i N_v(x)$$

$$H_v^{(2)}(x) = J_v(x) - i N_v(x)$$

It is often usefull to know the zeros of Bessel functions, x_{2n}

$$J_0(x_{2n}) = 0, \quad n = 1, 2, 3, \dots$$

as they naturally appear in Dirichlet problems as well as the zeros of the derivative of the Bessel functions, y_{2n}

$$\left. \frac{d J_0(x)}{d x} \right|_{x=y_{2n}} = 0, \quad n = 1, 2, 3, \dots$$

The normalization integral

$$\int_0^a d\rho \rho J_0(x_{2n} \frac{\rho}{a}) J_0(x_{2n} \frac{\rho}{a}) = \frac{a^2}{2} [J_{2n+1}(x_{2n})]^2 \delta_{nn}$$

allows a Fourier-Bessel expansion of any function $f(\rho)$ in the interval $0 \leq \rho \leq a$

$$f(\rho) = \sum_{n=1}^{\infty} A_{2n} J_0(x_{2n} \frac{\rho}{a}),$$

$$A_{2n} = \frac{2}{a^2 J_{2n+1}^2(x_{2n})} \int_0^a d\rho \rho f(\rho) J_0\left(\frac{x_{2n}\rho}{a}\right)$$

assuming without proof that $\{J_0(x_{2n} \frac{\rho}{a})\}_{n=1}^{\infty}$ form a complete set
Similar expansion holds with x_{2n} replaced by y_{2n} .

If we were choosing negative k^2 , so that instead of $Z'' - k^2 Z = 0 \Rightarrow Z'' + k^2 Z = 0$

we would have $Z = e^{\pm i k z}$ instead of $Z = e^{\pm k z}$

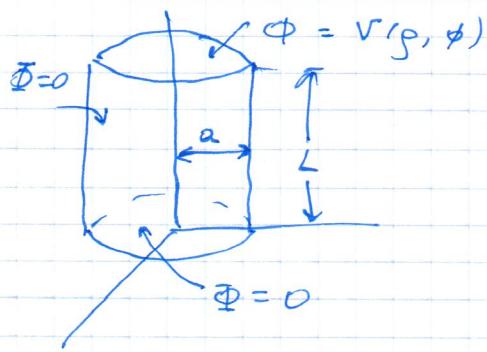
and modified Bessel equation instead of a regular one:

$$\frac{d^2 R}{dx^2} + \frac{1}{x} \frac{dR}{dx} - \left(1 + \frac{z^2}{x^2}\right) R = 0 \quad \text{with the solutions:}$$

$$I_0(x) = i^{-\frac{1}{2}} J_0(ix) \quad K_0(x) = \frac{\pi}{2} i^{\frac{1}{2}+1} H_0^{(1)}(ix) \quad \begin{cases} \text{Modified} \\ \text{Bessel} \\ \text{Functions.} \end{cases}$$

Ch. 3.8 Boundary-Value Problems in Cylindrical Coordinates

P. 11



$$\Phi = R(r) Q(\phi) Z(z)$$

$$Q(\phi) = A \sin m\phi + B \cos m\phi$$

$$Z(z) = \sinh kz$$

$$\Rightarrow R(r) = C J_m(kr) + D N_m(kr)$$

Φ is finite at $r \rightarrow 0 \Rightarrow D = 0$

$$\Phi = 0 \text{ at } r = a \Rightarrow k = k_{mn} = \frac{x_{mn}}{a}, n = 1, 2, 3, \dots$$

$$J_m(x_{mn}) = 0$$

$$\Rightarrow \Phi(r, \phi, z) = \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} J_m(k_{mn}r) \sinh(k_{mn}z) \times \\ \times (A_{mn} \sin m\phi + B_{mn} \cos m\phi)$$

B.C. at $z = L$ gives

$$V(r, \phi) = \sum_{mn} \sinh(k_{mn}L) J_m(k_{mn}r) (A_{mn} \sin m\phi + B_{mn} \cos m\phi)$$

We can find the expansion coefficient A_{mn} , B_{mn} in two steps. First according to Fourier theorem,

$$\frac{1}{\pi} \int_0^{2\pi} d\phi V(r, \phi) \sin m\phi = \sum_n \sinh(k_{mn}L) J_m(k_{mn}r) A_{mn} \quad m \neq 0$$

$$\frac{1}{\pi} \int_0^{2\pi} d\phi V(r, \phi) \cos m\phi = \sum_n \sinh(k_{mn}L) J_m(k_{mn}r) B_{mn} \quad m \neq 0$$

$$m=0 \quad \frac{1}{2\pi} \int_0^{2\pi} d\phi V(r, \phi) = \sum_n \sinh(k_{0n}L) J_0(k_{0n}r) (A_{0n} + B_{0n})$$

Now according to a Fourier-Bessel expansion we
considered before

$$A_{mn} = \frac{2[\sinh(K_{mn}L)]^{-1}}{\pi a^2 J_{m+1}^2(K_{mn}a)} \int_0^{2\pi} d\phi \int_0^a dp p V(p, \phi) J_m(K_{mn}p) \sin m\phi \quad \left. \right\} m \neq 0$$

$$B_{mn} = \frac{2[\sinh(K_{mn}L)]^{-1}}{\pi a^2 J_{m+1}^2(K_{mn}a)} \int_0^{2\pi} d\phi \int_0^a dp p V(p, \phi) J_m(K_{mn}p) \cos m\phi$$

For $m=0$

$$A_{0n} + B_{0n} = \frac{[\sinh(K_{mn}L)]^{-1}}{\pi a^2 J_{m+1}(K_{mn}a)} \int_0^{2\pi} d\phi \int_0^a dp p V(p, \phi) J_0(K_{mn}p)$$

Alternatively just use $\frac{1}{2} B_{0m}$ in all expansions
and use the same expressions as for $m \neq 0$.

Let's consider another example of the Dirichlet problem in $z \geq 0$ region : the potential Φ is finite in $z \geq 0$ and decays to zero for $z \rightarrow +\infty$.

The potential is specified on $z=0$ plane

$$\Phi(x, y, z=0) = \Phi(p, \phi, z=0) = V(p, \phi).$$

Then

$$\Phi(p, \phi, z) = \sum_{m=0}^{\infty} \int_0^{\infty} dk e^{-kz} J_m(kp) [A_m(k) \sin m\phi + B_m(k) \cos m\phi]$$

the B.C. demands

$$V(p, \phi) = \sum_{m=0}^{\infty} \int_0^{\infty} dk J_m(kp) [A_m(k) \sin m\phi + B_m(k) \cos m\phi]$$

(P.5)

$$\text{For } m \neq 0 \quad \frac{1}{\pi} \int_0^{2\pi} d\phi V(p, \phi) \begin{Bmatrix} \sin m\phi \\ \cos m\phi \end{Bmatrix} = \int_0^{\infty} dk' J_m(k' p) \begin{Bmatrix} A_m(k') \\ B_m(k') \end{Bmatrix}$$

P. 6

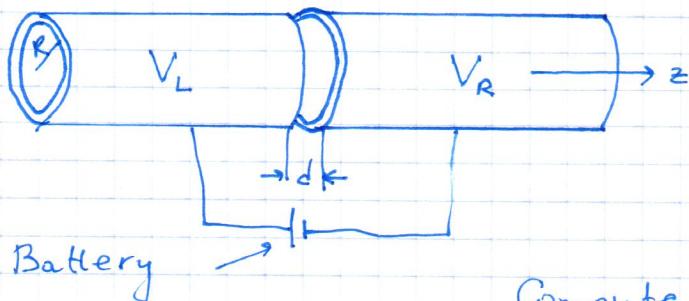
To find $A_m(k)$ and $B_m(k)$ use the relation

$$\int_0^{\infty} dx x J_m(kx) J_m(k'x) = \frac{1}{k'} \delta(k - k')$$

$$\Rightarrow \begin{Bmatrix} A_m(k) \\ B_m(k) \end{Bmatrix} = \frac{k}{\pi} \int_0^{\infty} dp p \int_0^{2\pi} d\phi V(p, \phi) J_m(kp) \begin{Bmatrix} \sin mp \\ \cos mp \end{Bmatrix} \quad m \neq 0$$

$$m=0: A_0(k) + B_0(k) = \frac{k}{2\pi} \int_0^{\infty} dp p \int_0^{2\pi} d\phi V(p, \phi) J_0(kp)$$

Application 7.4 (A.2.) : An Electrostatic Lens



Two adjacent and coaxial metal tubes of radius R separated by a small gap d . A potential difference $V_R - V_L$

is maintained between the tubes.

Compute the potential inside the tubes in the limit $d \rightarrow 0$.

$$\Phi(p, z) = \int_{-\infty}^{+\infty} \frac{dk}{2\pi} A(k) I_0(|k|p) e^{ikz}, \quad A(k) = ?$$

$$\Phi(p=R, z) = V_L \Theta(-z) + V_R \Theta(z) = \int_{-\infty}^{+\infty} \frac{dk}{2\pi} A(k) I_0(|k|R) e^{ikz}$$

$$\Rightarrow A(k) = \frac{1}{I_0(|k|R)} \left[V_L \int_{-\infty}^0 dz e^{-ikz} + V_R \int_0^{\infty} dz e^{-ikz} \right] = \frac{1}{ik} \frac{V_R - V_L}{I_0(|k|R)}$$

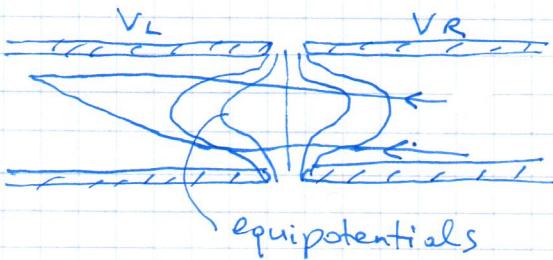
where without loss of generality we have omitted

the constant $\frac{V_L + V_R}{2}$ (alternatively set $\frac{V_L + V_R}{2} = 0$)

$$\Rightarrow \Phi(p, z) = \frac{V_R - V_L}{2\pi i} \int_{-\infty}^{+\infty} \frac{dk}{k} \frac{I_0(|k|p)}{I_0(|k|R)} e^{ikz} + \text{const}$$

Restoring ω constant and noticing that $I_0(k\rho) = I_0(k\rho)$, $I_0(1k/R) = I_0(kR)$ we obtain

$$\Phi(\rho, z) = \frac{V_R + V_L}{2} + \frac{V_R - V_L}{\pi} \int_0^\infty \frac{dk}{k} \frac{I_0(k\rho)}{I_0(kR)} \sin kz$$



Focusing properties



The force is focusing at the beginning and actually defocusing at the end (moving from left to right). As the particles (charged positively and moving from left to right with $V_L > V_R$) also accelerate the bending of the trajectory at the end is weaker than the bending of trajectories at the beginning. In other words : the initial focusing is more efficient than the final defocusing.

Note : The integrals over z that define $A(k)$ are ill-defined. They have to be regularized. Such a regularization must reflect the fact that the cylinders are of finite extent.

The convenient way to cut off the z -integrations is to consider the potential on the tubes :

$$\Phi_\epsilon(\rho=R, z) = V_L \Theta(-z) e^{\epsilon z} + V_R \Theta(z) e^{-\epsilon z}$$

Let's solve for $\Phi_\epsilon(p, z)$ and obtain $\Phi(p, z) = \lim_{\epsilon \rightarrow 0} \Phi_\epsilon(p, z)$

P.8

$$\Phi_\epsilon(p, z) = \int_{-\infty}^{+\infty} \frac{dk}{2\pi} A_\epsilon(k) I_0(|k|p) e^{ikz}$$

$$A_\epsilon(k) = \frac{1}{I_0(1k|R)} \left[V_L \int_{-\infty}^0 dz e^{-ikz + \epsilon z} + V_R \int_0^\infty dz e^{-ikz - \epsilon z} \right] =$$

$$= \frac{1}{I_0(1k|R)} \left[\frac{V_L + V_R}{2} \frac{2\epsilon}{k^2 + \epsilon^2} + \frac{V_R - V_L}{2} \frac{-2ik}{k^2 + \epsilon^2} \right]$$

$$\Phi_\epsilon(p, z) = \int_{-\infty}^{+\infty} \frac{dk}{2\pi} \frac{I_0(|k|p)}{I_0(1k|R)} \left[\frac{V_L + V_R}{2} \frac{2\epsilon}{k^2 + \epsilon^2} + \frac{V_R - V_L}{2} \frac{-2ik}{k^2 + \epsilon^2} \right] e^{ikz}$$

$\frac{\epsilon}{k^2 + \epsilon^2} \xrightarrow{\epsilon \rightarrow 0} \pi \delta(k)$, and as $\frac{k}{k^2 + \epsilon^2}$ is odd in k

$$\lim_{\epsilon \rightarrow 0} \Phi_\epsilon(p, z) = \Phi(p, z) = \int_{-\infty}^{+\infty} \frac{dk}{2\pi} \frac{I_0(1k|p)}{I_0(1k|R)} \frac{V_L + V_R}{2} 2\pi \delta(k) +$$

$$+ \lim_{\epsilon \rightarrow 0} \int \frac{dk}{2\pi} \frac{V_R - V_L}{2} \frac{-2ik}{k^2 + \epsilon^2} \frac{I_0(1k|p)}{I_0(1k|R)} i \sin kz$$

As the limit $\epsilon \rightarrow 0$ is well defined we get back to

$$\Phi(p, z) = \frac{V_L + V_R}{2} + \frac{V_R - V_L}{\pi} \int_0^\infty \frac{dk}{k} \frac{I_0(1k|p)}{I_0(1k|R)} \sin kz$$

Note that the way we chose to regularize the integral over z is immaterial. If we have a finite tube length, L as the regularization, the potential we would find is the same as long as at the point of observation, $|z| \ll \{L, 1/\epsilon\}$.

This can be shown explicitly, but is obvious on physical grounds.