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Chapter 3.9 J.D.J. Expansion of Green Functions in Spherical Coordinates

Green function $G(\vec{x}, \vec{x}')$ satisfies the Poisson equation

$$\nabla_{\vec{x}}^2 G(\vec{x}, \vec{x}') = -4\pi \delta(\vec{x} - \vec{x}')$$

with some BC (Dirichlet, Neumann)

If the surfaces at which the BC is specified are parametrized as $\beta_i \equiv \text{const}$ in some separable coordinate system $(\beta_1, \beta_2, \beta_3)$, the Green function is conveniently represented as sum of products of functions obtained by solving the Laplace equation in these coordinates.

Here we follow this route for spherical coordinates.

Example 1

Empty space (no boundaries, except for infinity where the potential is required to vanish)

$$G(\vec{x}, \vec{x}') \stackrel{\text{translational invariance}}{=} G(\vec{x} - \vec{x}') = \frac{1}{|\vec{x} - \vec{x}'|} = \sum_{l=0}^{\infty} \frac{r'_<^l}{r'_>^{l+1}} P_l(\cos \gamma),$$

where $P_l(x)$ is l -th Legendre polynomial

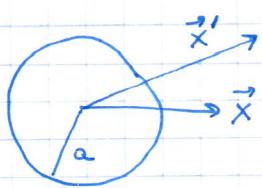
$$\cos \gamma = \hat{x} \cdot \hat{x}', \quad r'_< = \min\{r, r'\}, \quad r'_> = \max\{r, r'\}$$

$$\begin{aligned} \text{Addition Theorem } \Rightarrow G(\vec{x}, \vec{x}') &= 4\pi \sum_{l=0}^{\infty} \sum_{m=-l}^l \frac{1}{2l+1} \times \\ &\times \frac{r'_<^l}{r'_>} Y_m^*(\theta', \phi') Y_m(\theta, \phi) \end{aligned}$$

and this achieves the goal of expansion we are illustrating here.

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Example 2 : Exterior problem with a spherical boundary at $r=a$; Dirichlet Problem



Recall that we were looking for the Green Function in the form

$$G_D(\vec{x}, \vec{x}') = \frac{1}{|\vec{x} - \vec{x}'|} + F(\vec{x}, \vec{x}')$$

such that $\nabla_{\vec{x}}^2 F(\vec{x}, \vec{x}') = 0$ for all \vec{x}, \vec{x}' in the region of interest $\{|x|=r, |x'|=r'\} > a$.

We have found the function $F(\vec{x}, \vec{x}')$ by the method of images

$$G_D(\vec{x}, \vec{x}') = \frac{1}{|\vec{x} - \vec{x}'|} + \frac{q_{im}}{|\vec{x} - \vec{x}'_{im}|}$$

$$\vec{x}'_{im} = \vec{x}' \left(\frac{a}{r'} \right)^2, \quad q_{im} = -\frac{a}{r'}$$

$$G_D(\vec{x}, \vec{x}') = \frac{1}{|\vec{x} - \vec{x}'|} + \frac{-a/r'}{|\vec{x} - \vec{x}' \left(\frac{a}{r'} \right)^2|} = \\ = 4\pi \sum_{lm} \frac{1}{2l+1} \underbrace{\left[\frac{r'_<^e}{r'_>^{e+1}} - \frac{a}{r'}, \frac{(a/r')^e}{r'^{e+1}} \right]}_{\left[\frac{r'_<^l}{r'_>^{e+1}} - \frac{1}{a} \left(\frac{a^2}{rr'} \right)^{e+1} \right]} Y_{lm}^*(\theta', \phi') Y_{lm}(\theta, \phi)$$

$$\begin{cases} \frac{1}{r'^{e+1}} \left(r'^e - \frac{a^{2e+1}}{r'^{e+1}} \right), & r' > r \\ \frac{1}{r'^{e+1}} \left(r'^e - \frac{a^{2e+1}}{r'^{e+1}} \right), & r' < r \end{cases}$$

Note that $G_D = 0$ for $r' \rightarrow a$ and for $r \rightarrow a$

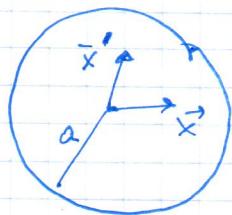
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(For $r \rightarrow a$ use the definition in the first line)

The prefactor that depends on r, r' is symmetric in $r \leftrightarrow r'$. Since $\sum_{m=-l}^l Y_m^*(\theta, \phi) Y_m(\theta', \phi') \propto \propto P_l(\cos(\theta - \theta'))$ there is a full symmetry $G_D(\vec{x}, \vec{x}') = G_D(\vec{x}', \vec{x})$ as expected.

The difference between $r' > r$ and $r' < r$ definitions is the result of the fact that $G_D(\vec{x}, \vec{x}')$ solves Poisson equation with the δ -function source.

Example 3 Interior Dirichlet problem inside the sphere $r=a$



$$\begin{aligned}
 G_D(\vec{x}, \vec{x}') &= \frac{1}{|\vec{x} - \vec{x}'|} - \frac{a/r'}{|\vec{x} - \vec{x}' (\frac{a}{r'})^2|} \\
 &= 4\pi \sum_{l=0}^{\infty} \frac{1}{2l+1} \left[\frac{r'^l}{r_s^{l+1}} - \frac{a}{r'} \frac{r^l}{r'^{l+1} (\frac{a}{r'})^{2(l+1)}} \right] Y_m^*(\theta, \phi) Y_m(\theta', \phi') \\
 &\quad \underbrace{\left[\frac{r'^l}{r_s^{l+1}} - \frac{1}{a} \frac{r^l r'^l}{a^{2l}} \right]}_{=} \\
 &\quad \begin{cases} .r'^l \left(\frac{1}{r^{l+1}} - \frac{r^l}{a^{2l+1}} \right) & r' < r \quad (\text{surely holds for } r \rightarrow a) \\ r^l \left(\frac{1}{r'^{l+1}} - \frac{r'^l}{a^{2l+1}} \right) & r' > r \quad (\text{surely holds for } r' \rightarrow a) \end{cases}
 \end{aligned}$$

④

Before we continue let's recall the few basic properties of spherical harmonics, Y_{lm}. First, they satisfy the following differential equation

$$\frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial Y_{lm}}{\partial \theta} \right) + \left[l(l+1) Y_{lm} + \frac{1}{\sin^2 \theta} \frac{\partial^2 Y_{lm}}{\partial \phi^2} \right] = 0$$

indeed $Y_{lm}(\theta, \phi) \propto P_l^m(\cos \theta) e^{im\phi}$ (up to a coefficient)

$$\Rightarrow \frac{\partial^2 Y_{lm}}{\partial \phi^2} \propto P_l^m(\cos \theta) (-m^2) e^{im\phi} = -m^2 Y_{lm}$$

so we have to show

$$\frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial Y_{lm}}{\partial \theta} \right) + \left[l(l+1) - \frac{m^2}{\sin^2 \theta} \right] Y_{lm} = 0$$

But after cancellation of a $e^{im\phi}$ factor it becomes just the defining equation for the Legendre (associated) polynomials $P_l^m(\cos \theta)$ and we are done.

The second property is completeness

$$\sum_{l=0}^{\infty} \sum_{m=-l}^l Y_{lm}^*(\theta, \phi) Y_{lm}(\theta', \phi') = \delta(\phi - \phi') \delta(\cos \theta - \cos \theta')$$

Now we are back to the defining equation (5)

$$\nabla_{\vec{x}}^2 G(\vec{x}, \vec{x}') = -4\pi \delta(\vec{x} - \vec{x}')$$

in spherical coordinates

Write

$$\delta(\vec{x} - \vec{x}') = \frac{1}{r^2} \delta(r - r') \delta(\phi - \phi') \delta(\cos \theta - \cos \theta') =$$

$$= \frac{1}{r^2} \delta(r - r') \sum_{l=0}^{\infty} \sum_{m=-l}^{l} Y_m^*(\theta' \phi') Y_m(\theta, \phi)$$

Completeness of Yem

Make an expansion $G(\vec{x}, \vec{x}')$ considering it a function of \vec{x} at given \vec{x}' (parameter)

$$G(\vec{x}, \vec{x}') = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} A_{lm}(r | r' \theta' \phi') Y_m(\theta, \phi)$$

\vec{x}' is parameter

The Laplacian in spherical coordinates

$$\nabla^2 \psi = \frac{1}{r^2} \frac{\partial^2}{\partial r^2} (r^2 \psi) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} (\sin \theta \frac{\partial \psi}{\partial \theta}) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 \psi}{\partial \phi^2}$$

Using the properties of spherical harmonics

$$\begin{aligned} & \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \frac{1}{r} \frac{\partial^2}{\partial r^2} r A_{lm}(r | r' \theta' \phi') Y_m(\theta, \phi) \\ & + \frac{(-1)}{r^2} (l)(l+1) A_{lm}(r | r' \theta' \phi') Y_m(\theta, \phi) = \\ & = -4\pi \frac{1}{r^2} \delta(r - r') \sum_{l=0}^{\infty} \sum_{m=-l}^{l} Y_m^*(\theta' \phi') Y_m(\theta, \phi) \end{aligned}$$

Y_m are linearly independent \Rightarrow

For all l and m

$$\frac{1}{r} \frac{\partial^2}{\partial r^2} (r A_{lm}(r|r'\theta'\phi')) - \frac{l(l+1)}{r^2} A_{lm}(r|r'\theta'\phi') = \\ = -\frac{4\pi}{r^2} \delta(r-r') Y_m^*(\theta'\phi')$$

It follows that the expansion of $A_{lm}(r|r'\theta'\phi')$ in $Y_m^*(\theta'\phi')$ contains just one term:

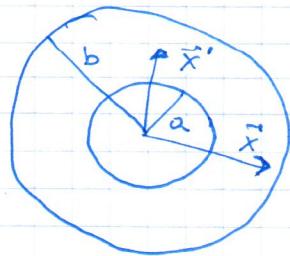
$$A_{lm}(r|r'\theta'\phi') = g_e(r, r') Y_m^*(\theta'\phi') , \text{ and}$$

$$\frac{1}{r} \frac{d^2}{dr^2} (r g_e(r, r')) - \frac{l(l+1)}{r^2} g_e(r, r') = -\frac{4\pi}{r^2} \delta(r-r') \quad (*)$$

The general solution at $r \neq r'$ reads

$$g_e(r, r') = \begin{cases} Ar^l + Br^{-l-1} & r < r' \\ A'r^l + B'r^{-l-1} & r > r' \end{cases}$$

Specify to the problem of Dirichlet in between the two concentric spheres. From B.C.:



$$g_e(r, r') = \begin{cases} A(r^l - \frac{a^{2l+1}}{r^{l+1}}) & r < r' \\ B'(\frac{1}{r^{l+1}} - \frac{r^l}{b^{2l+1}}) & r > r' \end{cases}$$

$$g_e(r, r') = g_e(r', r) \Rightarrow$$

$$g_e(r, r') = C \left(r_<^l - \frac{a^{2l+1}}{r_<^{l+1}} \right) \left(\frac{1}{r_>^{l+1}} - \frac{r_>^l}{b^{2l+1}} \right)$$

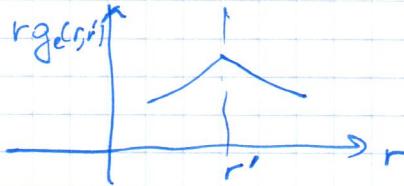
To fix C apply the operation $\lim_{\epsilon \rightarrow 0} \int_{r'-\epsilon}^{r'+\epsilon} dr r^n$ to both sides of $(*)$ to

obtain the matching condition:

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$$\left\{ \frac{d}{dr} [r g_e(r, r')] \right\}_{r' + \epsilon} - \left\{ \frac{d}{dr} [r g_e(r, r')] \right\}_{r' - \epsilon} = -\frac{4\pi}{r'}$$

$\Rightarrow g_e(r, r')$ suffers cusp non-analyticity at $r = r'$ which is a consequence of the presence of a charge at $r = r'$.



$$\left\{ \frac{d}{dr} [r g_e(r, r')] \right\}_{r' + \epsilon} \stackrel{r > r'}{\downarrow} = C \left(r'^{\ell} - \frac{a^{2\ell+1}}{r'^{\ell+1}} \right) \left[\frac{1}{dr} \left(\frac{1}{r^{\ell}} - \frac{r^{\ell+1}}{b^{2\ell+1}} \right) \right]_{r=r'} \\ = -\frac{C}{r'} \left[1 - \left(\frac{a}{r'} \right)^{2\ell+1} \right] \left[\ell + (\ell+1) \left(\frac{r'}{b} \right)^{2\ell+1} \right]$$

(A)

Similarly

$$\left\{ \frac{d}{dr} [r g_e(r, r')] \right\}_{r=r'-\epsilon} \stackrel{r < r'}{\downarrow} = C \left[\frac{1}{dr} \left(r^{\ell+1} - \frac{a^{2\ell+1}}{r^{\ell}} \right) \right]_{r=r'} \times \left(\frac{1}{r'^{\ell+1}} - \frac{r'^{\ell}}{b^{2\ell+1}} \right) \\ = C \left[(\ell+1) r'^{\ell} + \ell \frac{a^{2\ell+1}}{r'^{\ell+1}} \right] \left[\frac{1}{r'^{\ell+1}} - \frac{r'^{\ell}}{b^{2\ell+1}} \right] = \\ = \frac{C}{r'} \left[(\ell+1) r'^{\ell+1} + \ell \frac{a^{2\ell+1}}{r'^{\ell}} \right] \left(\frac{1}{r'^{\ell+1}} - \frac{r'^{\ell}}{b^{2\ell+1}} \right) = \\ = \frac{C}{r'} \left[(\ell+1) + \ell \left(\frac{a}{r'} \right)^{2\ell+1} \right] \left(1 - \left(\frac{r'}{b} \right)^{2\ell+1} \right)$$

(B)

Let's compare expressions (A) and (B).

power $(r')^{-(2\ell+2)}$: (A) $\Rightarrow \frac{C\ell}{r'} \left(\frac{a}{r'} \right)^{2\ell+1}$ \searrow cancel out
 (B) $\Rightarrow \frac{C\ell}{r'} \left(\frac{a}{r'} \right)^{2\ell+1} \nearrow$

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power $(r')^{2\ell}$

$$(A) \Rightarrow -\frac{C}{r'} (\ell+1) \left(\frac{r'}{b}\right)^{2\ell+1}$$

$$(B) \Rightarrow -\frac{C}{r'} (\ell+1) \left(\frac{r'}{b}\right)^{2\ell+1} \quad \cancel{\text{cancel out}}$$

power $(r')^{-1}$

$$(A) \Rightarrow -\frac{C}{r'} \left[\ell - (\ell+1) \left(\frac{a}{b}\right)^{2\ell+1} \right]$$

$$(B) \Rightarrow \frac{C}{r'} \left[\ell+1 - \ell \left(\frac{a}{b}\right)^{2\ell+1} \right]$$

Matching condition:

$$-\frac{C}{r'} \left[(2\ell+1) - (2\ell+1) \left(\frac{a}{b}\right)^{2\ell+1} \right] = -\frac{4\pi}{r'}$$

$$\Rightarrow C = \frac{4\pi}{(2\ell+1) \left[1 - \left(\frac{a}{b}\right)^{2\ell+1} \right]}$$

$$\Rightarrow G_D(\bar{x}, \bar{x}') = 4\pi \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \frac{Y_m^*(\theta' \phi') Y_m(\theta, \phi)}{(2\ell+1) \left[1 - \left(\frac{a}{b}\right)^{2\ell+1} \right]} \times$$

$$\times \left(r'_<^\ell - \frac{a^{2\ell+1}}{r'_<} \right) \left(\frac{1}{r'_>^{2\ell+1}} - \frac{r'_>^\ell}{b^{2\ell+1}} \right)$$

Limit $a \rightarrow 0 \Rightarrow$ obtain the previous result for the interior Dirichlet problem

$$\text{NOTE: } r'_<^\ell r'_>^\ell = r_r^\ell r_l^\ell$$

Limit $b \rightarrow \infty \Rightarrow$ obtain the previous result for the exterior Dirichlet problem

$$\text{NOTE: } \frac{1}{r'_>^{2\ell+1}} \frac{1}{r'_<} = \frac{1}{r_r^{2\ell+1} r_l^{\ell+1}}$$