

# Boundary-Value Problems in Electrostatics (Part I)

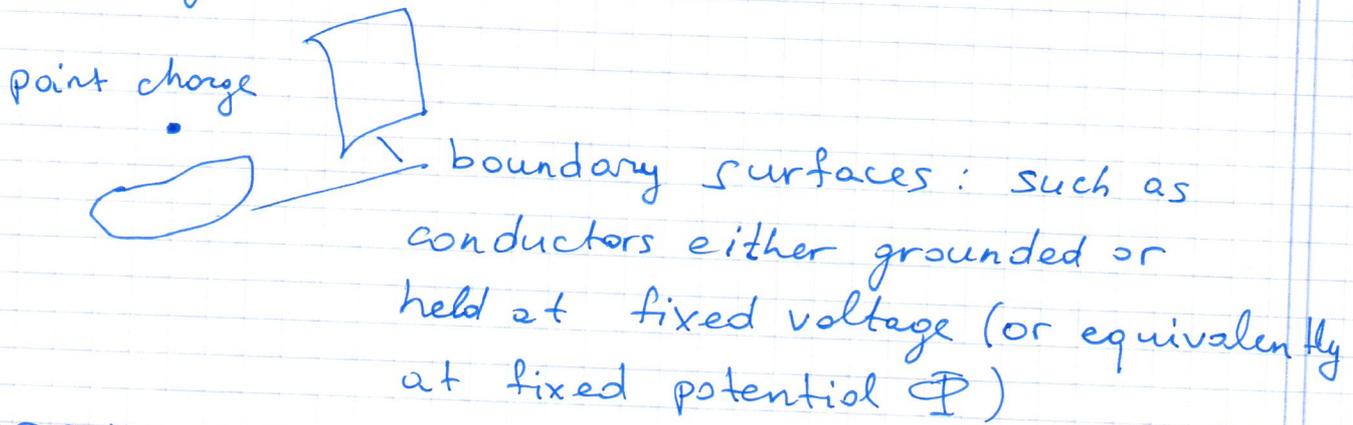
(1)

We start with the discussion of the two methods used to solve Boundary Value Problems

- (1) Method of Images (Closely related to the construction of Green functions)
- (2) expansion in orthogonal functions

## Ch. 2.1 J.D.J. Method of Images

Typical setting for the application of Method of Images

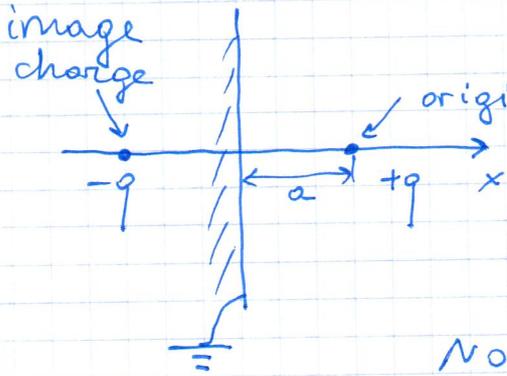


IDEA: Try to find a limited (finite) # of point charges of appropriate magnitude placed outside the region of interest to simulate the Boundary Conditions (BC). These charges are fictitious and are referred to as "image" charges.

METHOD: Remove the boundary surface and replace the effect of the charges on them by the potential produced by the image charges.

The image charges must be outside the region of interest because of the Poisson Equation.

A familiar example: let the point charge  $q$  be to the right of the  $x=0$  plane. The problem: find the potential  $\Phi$ , solving the Poisson Equation in  $x > 0$  region (that is our region of interest) such that  $\Phi(x=0) = 0$  (that is our BC)



$$\Phi = \Phi_q + \Phi_{-q} \quad \leftarrow \begin{array}{l} \text{image charge} \\ \text{contribution} \end{array}$$

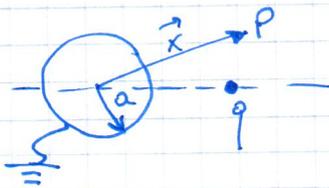
$$\Phi_q(\vec{x}) = \frac{1}{4\pi\epsilon_0} \frac{q}{|\vec{x} - \hat{x}a|}$$

$$\Phi_{-q} = -\frac{1}{4\pi\epsilon_0} \frac{q}{|\vec{x} + \hat{x}a|}$$

Now  $\nabla^2 \Phi = -\frac{1}{\epsilon_0} q \delta(\vec{x} - \hat{x}a)$  in  $\underline{x > 0}$  ✓

$$\Phi(x=0) = 0 \quad \checkmark$$

Ch. 2.2 J.D.J. Point Charge in the Presence of a Grounded Conducting Sphere



The point charge  $q$  is located outside the grounded sphere of radius  $a$ .

Problem: solve Poisson Equation in the region of interest  $|\vec{x}| > a$  (outside the sphere)

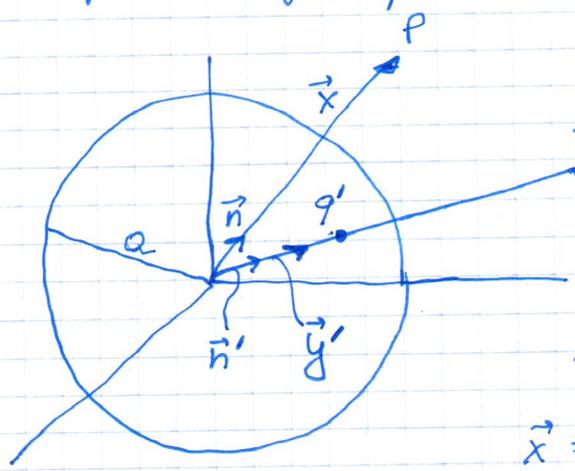
with BC  $\Phi(|\vec{x}|=a) = 0$

$\Rightarrow$  It follows that the image charge(s) has (ve) to be inside the sphere.

In addition by symmetry these point-charges must lie on the line combining the charge  $q$  and the center of the sphere.

Let's try to solve the problem with a single image charge  $q'$

(3)



- 1) The external charge  $q$  is located at  $\vec{y}$
  - 2) The image charge  $q'$  is located at  $\vec{y}'$
- $\Rightarrow \vec{y} \parallel \vec{y}'$ , and  $|\vec{y}'| < a$

Introduce unit vectors  $\vec{n}, \vec{n}'$  by  
 $\vec{x} = x\vec{n}, \vec{y} = y\vec{n}, \vec{y}' = y'\vec{n}'$

(\*)

Coulomb Law  $\Rightarrow \Phi(\vec{x}) = \frac{q/4\pi\epsilon_0}{|\vec{x} - \vec{y}|} + \frac{q'/4\pi\epsilon_0}{|\vec{x} - \vec{y}'|}$

or with notations (\*)

$$\Phi(\vec{x}) = \frac{q/4\pi\epsilon_0}{|x\vec{n} - y\vec{n}|} + \frac{q'/4\pi\epsilon_0}{|x\vec{n} - y'\vec{n}'|}$$

Let's try to find  $q'$  and  $y'$  such that

$$0 = \Phi(|\vec{x}| = a) \Rightarrow |a\vec{n} - y\vec{n}|q' + |a\vec{n} - y'\vec{n}'|q = 0$$

$\Rightarrow \text{sign } q' = -\text{sign } q$

(1)

$$(1) \Rightarrow q'^2(a^2 + y^2 - 2ay\vec{n} \cdot \vec{n}') = q^2(a^2 + y'^2 - 2ay'\vec{n} \cdot \vec{n}')$$

$$\Rightarrow q'^2(a^2 + y^2) = q^2(a^2 + y'^2) \text{ and } \boxed{q'^2 y = q^2 y'}$$

(2)

$$\frac{q'^2}{q^2}(a^2 + y^2) = (a^2 + y'^2) = \frac{y'}{y}(a^2 + y^2) = (a^2 + y'^2)$$

$$\Rightarrow y'^2 - y' \left( \frac{a^2}{y} + y \right) + a^2 = 0$$

$$\Rightarrow y' = \frac{1}{2} \left[ \frac{a^2}{y} + y \pm \sqrt{\left( \frac{a^2}{y} + y \right)^2 - 4a^2} \right]$$

$\underbrace{\hspace{10em}}_{\left( \frac{a^2}{y} - y \right)^2}$

So we have two possibilities

$$y' = \frac{1}{2} \left[ \frac{a^2}{y} + y \pm \left( \frac{a^2}{y} - y \right) \right]$$

$$1) + \Rightarrow y' = \frac{1}{2} \left[ \frac{a^2}{y} + y + \frac{a^2}{y} - y \right] = \frac{a^2}{y}$$

is legitimate as in this case  $y' < a$  and the image charge is outside the region of interest

$$2) - \Rightarrow y' = \frac{1}{2} \left[ \frac{a^2}{y} + y - \frac{a^2}{y} + y \right] = y$$

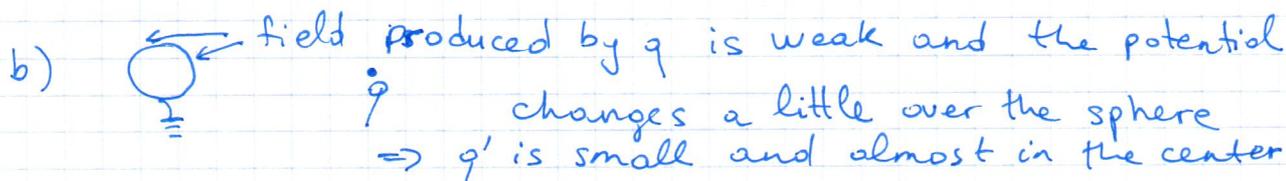
is impossible as then the image charge is inside the region of interest

So we take  $y' = \frac{a^2}{y}$ , Eq. (2)  $\Rightarrow q' = -q \frac{a}{y}$

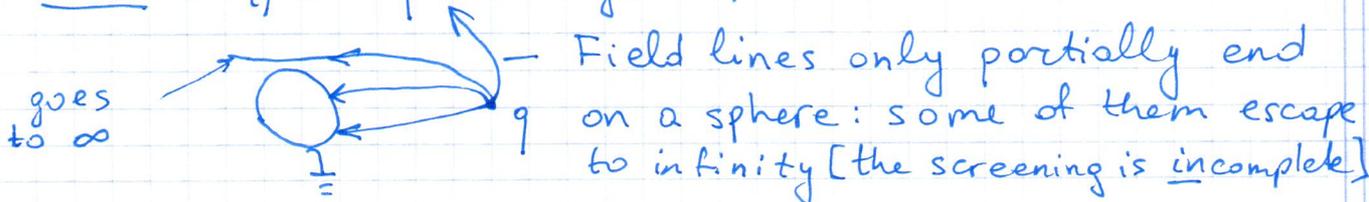
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in principle one still has to check that (3) indeed gives Eq. (1) which is easily done.

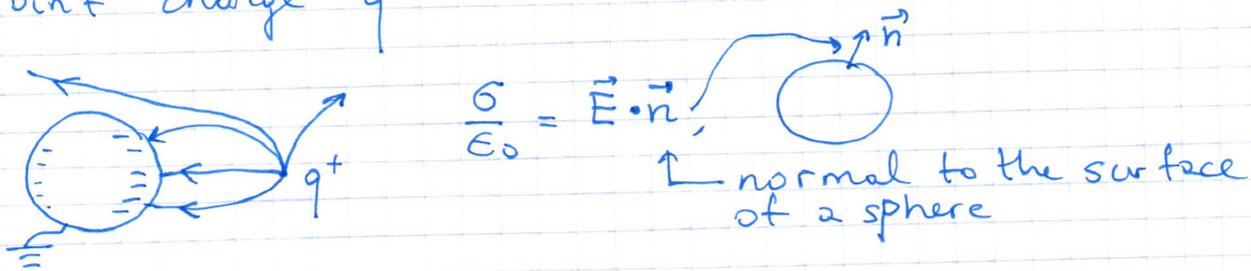
- Limiting cases: a)  $y \rightarrow a^+ \Rightarrow q' \rightarrow -q^-, y' \rightarrow a^-$
- b)  $y \rightarrow \infty \Rightarrow q' \rightarrow 0, y' \rightarrow 0$



Note  $|q'| < q$  always: explanation:



Let's compute the surface-charge density on a grounded sphere induced by the external point charge  $q$



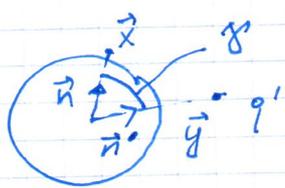
$$4\pi\epsilon_0\Phi(\vec{x}) = \frac{q}{|\vec{x}\vec{n} - y\vec{n}'|} + \frac{q'}{|\vec{x}\vec{n} - y'\vec{n}'|}$$

Recall  $yy' = a^2, q'y = -qa$

$$\vec{E}|_{x=a} = - \frac{\partial \Phi}{\partial \vec{x}}|_{x=a} \Rightarrow \vec{E} \cdot \vec{n} = - \frac{\partial \Phi}{\partial x}|_{x=a}$$

vector                      scalar

Explanation  $\vec{E} \cdot \vec{n} = - \nabla \Phi \cdot \vec{n} = - \lim_{\Delta x \rightarrow 0} \frac{1}{\Delta x} [\Phi(\vec{x} + \Delta x \vec{n}) - \Phi(\vec{x} \vec{n})] = - \frac{\partial \Phi}{\partial x}$



$$|\vec{x}\vec{n} - y\vec{n}'| = \sqrt{x^2 + y^2 - 2xy \cos \delta}$$

$$|\vec{x}\vec{n} - y'\vec{n}'| = \sqrt{x^2 + y'^2 - 2xy' \cos \delta}$$

Derivatives  $\frac{\partial}{\partial x} \frac{1}{|\vec{x}\vec{n} - y\vec{n}'|} = -\frac{1}{2} \frac{2x - 2y \cos \delta}{(x^2 + y^2 - 2xy \cos \delta)^{3/2}}$

$$\frac{\partial}{\partial x} \frac{1}{|\vec{x}\vec{n} - y'\vec{n}'|} = -\frac{1}{2} \frac{2x - 2y' \cos \delta}{(x^2 + y'^2 - 2xy' \cos \delta)^{3/2}}$$

Using (3)

$$-\frac{4\pi\epsilon_0}{q} \frac{\partial \Phi}{\partial x}|_{x=a} = \frac{a - y \cos \delta}{(a^2 + y^2 - 2ay \cos \delta)^{3/2}} - \frac{a}{y} \frac{a - y' \cos \delta}{(a^2 + y'^2 - 2ay' \cos \delta)^{3/2}}$$

contribution of a real charge  $q$                       contribution of an image charge  $q'$

(6)

Let's rewrite denominators:

$$(a^2 + y^2 - 2ay \cos \delta)^{3/2} = a^3 \left(1 + \frac{y^2}{a^2} - 2 \frac{y}{a} \cos \delta\right)^{3/2}$$

$$(a^2 + y'^2 - 2ay' \cos \delta)^{3/2} = y'^3 \left(\frac{a^2}{y'^2} + 1 - 2 \frac{a}{y'} \cos \delta\right)^{3/2} \quad \text{Eq. (3)}$$

$$= y'^3 \left(\frac{y^2}{a^2} + 1 - 2 \frac{y}{a} \cos \delta\right)^{3/2}$$

Using this form of denominators we get

$$-\frac{4\pi\epsilon_0}{9} \frac{\partial \Phi}{\partial x} \Big|_{x=a} = \frac{1}{\left(1 + \frac{y^2}{a^2} - 2 \frac{y}{a} \cos \delta\right)^{3/2}} \left[ \frac{1}{a^3} (a - y \cos \delta) - \frac{a}{y y'^3} (a - y' \cos \delta) \right]$$

Rewrite the square brackets to exclude  $y'$

$$\left[ \right] = \frac{1}{a^2} - \frac{a^2}{y y'^3} - \cos \delta \left( \frac{y}{a^3} - \frac{a}{y y'^3} \right)$$

Notice:

$$1) \frac{a}{y y'^2} = \frac{a y}{y^2 y'^2} = \frac{a y}{a^4} = \frac{y}{a^3} \Rightarrow \boxed{\frac{y}{a^3} - \frac{a}{y y'^2} = 0}$$

$$2) \frac{a^2}{y y'^3} = \frac{a^2 y^2}{y^3 y'^3} = \frac{a^2 y^2}{a^6} = \frac{y^2}{a^4} \Rightarrow \boxed{\frac{1}{a^2} - \frac{a^2}{y y'^3} = \frac{1}{a^2} - \frac{y^2}{a^4}}$$

Using 1) and 2)

$$\left[ \right] = \frac{1}{a^2} - \frac{y^2}{a^4} \Rightarrow$$

$$-\frac{4\pi\epsilon_0}{9} \frac{\partial \Phi}{\partial x} \Big|_{x=a} = \frac{1}{a^2} \frac{1 - y^2/a^2}{\left(1 + \frac{y^2}{a^2} - 2 \frac{y}{a} \cos \delta\right)^{3/2}} \quad \text{Eq. (4)}$$

is expressed as a function of a parameter  $y/a$

Parameter  $y/a > 1$ .

Alternative parameter is  $a/y < 1$  can sometimes be useful.

To express the result for  $\frac{\partial \Phi}{\partial x} \Big|_{x=a}$  via  $a/y$  rewrite the denominator as

$$(1 + \frac{y^2}{a^2} - 2 \frac{y}{a} \cos \delta)^{3/2} = \frac{y^3}{a^3} \left( \frac{a^2}{y^2} + 1 - 2 \frac{a}{y} \cos \delta \right)^{3/2}$$

and note that the numerator becomes

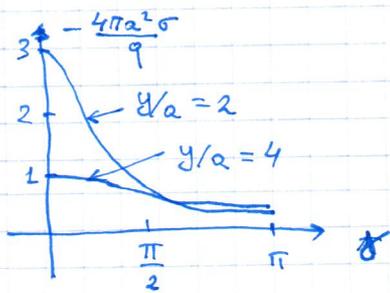
$$\frac{a^3}{y^3} \left( \frac{1}{a^2} - \frac{y^2}{a^4} \right) = \frac{a}{y} \left( \frac{1}{y^2} - \frac{1}{a^2} \right) = \frac{1}{a^2} \frac{a}{y} \left( \frac{a^2}{y^2} - 1 \right)$$

then the alternative form of Eq. (4) is

$$-\frac{4\pi\epsilon_0}{9} \frac{\partial \Phi}{\partial x} \Big|_{x=a} = \frac{1}{a^2} \left( \frac{a}{y} \right) \frac{a^2/y^2 - 1}{\left( \frac{a^2}{y^2} + 1 - 2 \frac{a}{y} \cos \delta \right)^{3/2}} \tag{Eq. 5}$$

Using either (4) or (5) we get for  $\sigma = -\epsilon_0 \frac{\partial \Phi}{\partial x} \Big|_{x=a}$

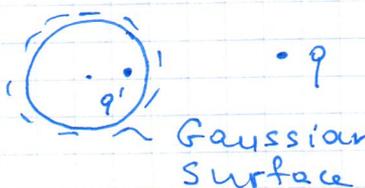
$$\sigma = \frac{q}{4\pi a^2} \frac{1 - y^2/a^2}{\left( 1 + \frac{y^2}{a^2} - 2 \frac{y}{a} \cos \delta \right)^{3/2}} = \frac{q}{4\pi a^2} \frac{a}{y} \frac{a^2/y^2 - 1}{\left( \frac{a^2}{y^2} + 1 - 2 \frac{a}{y} \cos \delta \right)^{3/2}} \tag{Eq. 6}$$



The total induced charge on a sphere is  $q'$ .

□ Use Gauss Law with the choice of the Gauss

surface being the sphere concentric with the conductor and having a radius  $a + 0^+$

  $\frac{Q}{\epsilon_0} = \oint \vec{E} \cdot d\vec{S} = \oint (\vec{E}_q + \vec{E}_{\text{induced}}) \cdot d\vec{S}$   
induced charge included in the Gaussian surface

where  $\vec{E}_q$  is the field due to the external charge  $q$ , and  $\vec{E}_{\text{induced}}$  is the field due

to the screening charges induced on a surface of a sphere.

Note 1) as the charge  $q$  is not within the Gaussian surface  $\oint \vec{E}_q \cdot d\vec{S} = 0$

2) The field  $\vec{E}_{induced}$  on a Gaussian surface exactly equals the field due to the image charge  $q'$ ,  $\vec{E}_{q'}$

$\Rightarrow \frac{Q}{\epsilon_0} = \oint \vec{E}_{q'} \cdot d\vec{S} = \frac{q'}{\epsilon_0} \Rightarrow \boxed{Q = q'}$  Eq. (7)

Let's make sure that Eq. (6) and Eq. (7) are consistent.



$Q = \oint da \sigma(\theta) = a^2 \int d\Omega \sigma(\cos\theta) = a^2 \int_{-\pi}^{\pi} d\phi \int_0^{\pi} d\theta \sin\theta \sigma(\cos\theta) = 2\pi a^2 \int_{-1}^1 d\mu \sigma(\mu), \mu \equiv \cos\theta$

From Eq. (6)  $\sigma(\mu) = \frac{C}{(A - B\mu)^{3/2}}$

Use auxiliary integral

$\int_{-1}^1 d\mu \frac{C}{(A - B\mu)^{3/2}} = \frac{2C}{B} \left[ \frac{1}{(A - B\mu)^{1/2}} \right]_{-1}^1 = \frac{2C}{B} \left[ \frac{1}{\sqrt{A-B}} - \frac{1}{\sqrt{A+B}} \right]$

Eq. (6)  $\Rightarrow Q = 2\pi a^2 \frac{-q}{4\pi a^2} \frac{a}{y} \left(1 - \frac{a^2}{y^2}\right) \int_{-1}^1 d\mu \frac{1}{\left(\frac{a^2}{y^2} + 1 - 2\frac{a}{y}\mu\right)^{3/2}}$

$= -\frac{qa}{2y} \left(1 - \frac{a^2}{y^2}\right) \frac{2}{2a} \left[ \frac{1}{\sqrt{\frac{a^2}{y^2} + 1 - 2\frac{a}{y}}} - \frac{1}{\sqrt{\frac{a^2}{y^2} + 1 + 2\frac{a}{y}}} \right]$

where we used the auxiliary integral with  $B = 2\frac{a}{y}, A = \frac{a^2}{y^2} + 1, C = 1$

Continue by writing

$$\frac{a^2}{y^2} + 1 \mp 2\frac{a}{y} = \left(\frac{a}{y} \mp 1\right)^2$$

$$y > a \Rightarrow \frac{a}{y} - 1 < 0 \Rightarrow \sqrt{\frac{a^2}{y^2} + 1 \mp 2\frac{a}{y}} = 1 \mp \frac{a}{y}$$

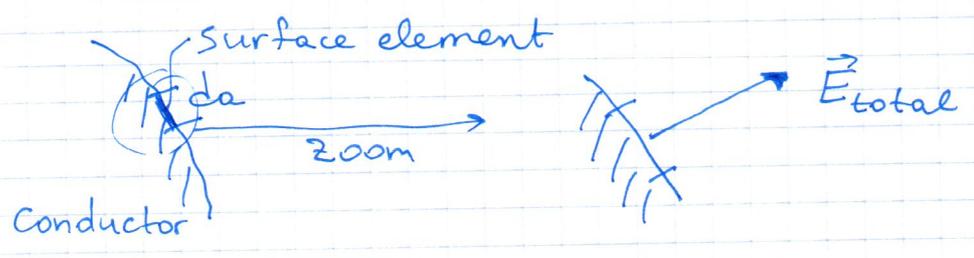
$$\Rightarrow Q = -\frac{q}{2} \left(1 - \frac{a^2}{y^2}\right) \left[\frac{1}{1 - \frac{a}{y}} - \frac{1}{1 + \frac{a}{y}}\right] = -q \frac{a}{y}$$

$\Rightarrow Q = q'$  in agreement with Gauss Law!

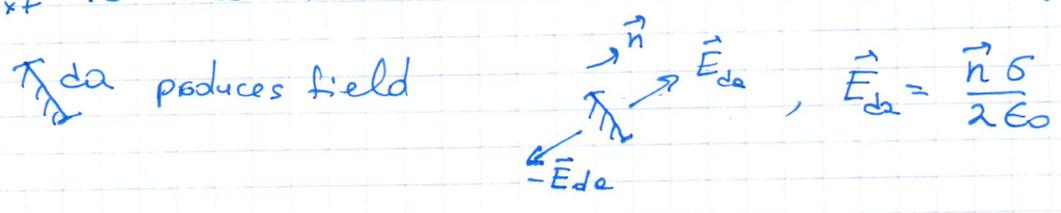
Let's also consider the force acting on a sphere. Coulomb Law complies with Newton 3rd Law so it should be the same (and oppositely directed) as the force acting on  $q$ :

$$|\vec{F}_q| = \frac{1}{4\pi\epsilon_0} q \left(q \frac{a}{y}\right) \frac{1}{\left(y - \frac{a^2}{y}\right)^2} = \frac{1}{4\pi\epsilon_0} \frac{q^2}{a^2} \left(\frac{a}{y}\right)^3 \left(1 - \frac{a^2}{y^2}\right)^{-2} \quad \text{Eq (8)}$$

Now let's evaluate the force acting on a sphere



$\vec{E}_{total} = \vec{E}_{da} + \vec{E}_{ext}$ ,  $\vec{E}_{da}$  is the field created by the charges in the surface element  $da$ .  $\vec{E}_{ext}$  is the field due to rest of the charges



But there is no field inside conductor

$$\vec{E}_{\text{ext}} - \vec{E}_{\text{da}} = 0 \Rightarrow \vec{E}_{\text{ext}} = \frac{\vec{n} \sigma}{2 \epsilon_0} \Rightarrow$$

$$d\vec{F} = \sigma da \vec{E}_{\text{ext}} = \frac{\vec{n} da \sigma^2}{2 \epsilon_0}$$


 $\vec{F} = |\vec{F}| \hat{z} \Rightarrow |\vec{F}| = F_z$

$\Rightarrow |\vec{F}| = \oint da \frac{\sigma^2}{2 \epsilon_0} \cos \theta$

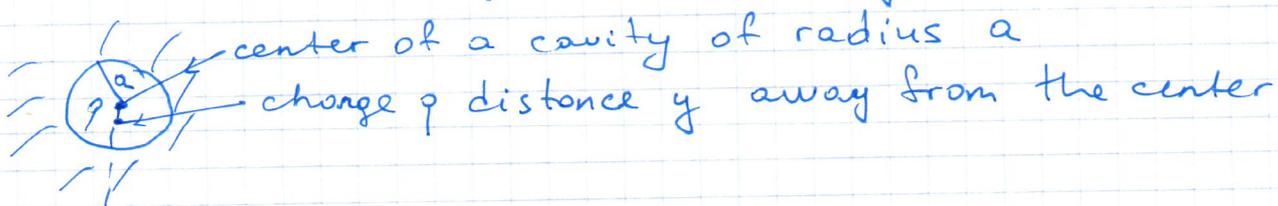
From projection on the  $\hat{z}$ -axis (axis of symmetry)

$$\Rightarrow |\vec{F}| = \frac{2\pi a^2}{2 \epsilon_0} \int_{-1}^1 d\mu \sigma^2(\mu) \mu = \left| \text{using Eq. 6} \right| =$$

$$= \frac{\pi a^2}{\epsilon_0} \frac{q^2}{16\pi^2 a^4} \left(\frac{a}{y}\right)^2 \left(1 - \frac{a^2}{y^2}\right)^2 \int_{-1}^1 d\mu \frac{\mu}{\left(1 + \frac{a^2}{y^2} - 2\frac{a}{y}\mu\right)^3}$$

The  $\mu$ -integration is easily done using Mathematica resulting in Eq. 8 as was expected from the 3-rd Newton Law.

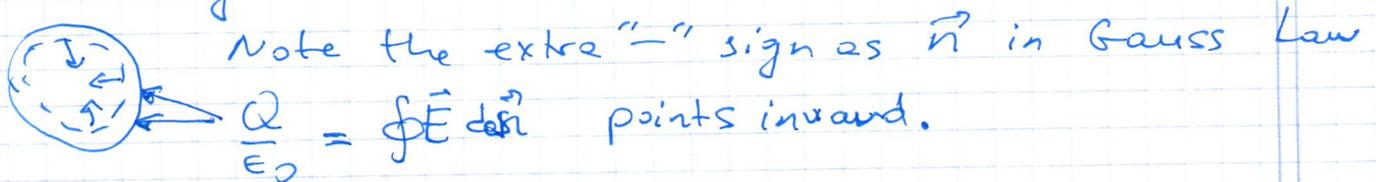
Consider a charge in the cavity



$$q' = -\frac{y'}{a} q, |q'| > |q|$$

The image charge  $q'$  at distance  $y' = \frac{a^2}{y}$

The total induced charge is  $-q$  as the Gaussian surface is the sphere that is concentric with the cavity with the radius  $a - 0^+$ .



### J.D.J. 2.3 Point Charge in the Presence of a Charged, Insulated, Conducting Sphere

Imagine that the total charge of a sphere is fixed to be  $Q$ , and the sphere is disconnected. To obtain the solution consider again grounded sphere we have found that the induced charge is  $q' = -aq/y$ , where  $y$  is again the distance from  $q$  to the center of a sphere

1)  the sphere is an equipotential surface (actually zero potential) } here total induced charge is  $q' = -aq/y$

2)  put evenly distributed over the surface charge  $Q + aq/y$

important: the sphere is again equi-potential surface (actually with the potential of a point-charge  $Q + aq/y$  at the center)

3) The superposition of two solutions 1) and 2) solves our problem as the surface of the sphere is again an equi-potential which means that charges on it are at equilibrium.

 
$$\Phi(\vec{x}) = \frac{1}{4\pi\epsilon_0} \left[ \frac{q}{|\vec{x} - \vec{y}|} - \frac{aq}{y|\vec{x} - \vec{y} \frac{a^2}{y^2}|} + \frac{Q + \frac{a}{y}q}{|\vec{x}|} \right] \quad |\vec{x}| > a$$

Solution of the grounded sphere problem

Let's compute the force on an external charge  $q$

(12)

$$\vec{F} = \frac{1}{4\pi\epsilon_0} \frac{\vec{y}}{y} \left[ \frac{q'q}{|y-\vec{y}'|^2} + \frac{(Q-q')q}{|y|^2} \right] =$$

$$= \frac{q}{4\pi\epsilon_0} \frac{\vec{y}}{y} \left[ \frac{Q}{|y|^2} - q \frac{a}{y} \left( \frac{1}{|y-\vec{y}'|^2} - \frac{1}{|y|^2} \right) \right]$$

Recall  $\vec{y}' = \vec{y} \frac{a^2}{y^2}$ , and  $|\vec{y}'| = y$

$$\Rightarrow \frac{1}{|y-\vec{y}'|^2} - \frac{1}{|y|^2} = \frac{1}{\left(y - y \frac{a^2}{y^2}\right)^2} - \frac{1}{y^2} = \frac{y^2}{(y^2 - a^2)^2} - \frac{1}{y^2} =$$

$$= \frac{y^4 - y^4 + 2y^2a^2 - a^4}{y^2(y^2 - a^2)^2} = \frac{a^2(2y^2 - a^2)}{y^2(y^2 - a^2)^2}$$

$$\Rightarrow \vec{F} = \frac{q}{4\pi\epsilon_0} \frac{1}{y^2} \frac{\vec{y}}{y} \left[ Q - \frac{qa^3(2y^2 - a^2)}{y(y^2 - a^2)^2} \right]$$

Limiting cases

1)  $y \gg a$   $\vec{F} \approx \frac{qQ}{4\pi\epsilon_0} \frac{1}{y^2} \frac{\vec{y}}{y}$  as expected

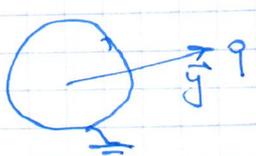
as screening (induced charges) are negligible

2)  $y \rightarrow a + 0^+$ ,  $F \approx -\frac{q^2}{4\pi\epsilon_0} \frac{1}{(2(y-a))^2} \frac{\vec{y}}{y}$

Explanation: the force is dominated by the screening (image) charge as the distance between  $q$  and  $q'$  becomes  $2(y-a)$  and  $q' \approx -q$

$\Rightarrow$  Conclusion: it is hard to remove charge off the conductor as at short distances there is a strong attraction to the conductor regardless of a total charge on a sphere.

Green function for the Sphere; General solution for the potential



$$\Rightarrow \Phi_{\vec{y}}(\vec{x}) = \frac{q/4\pi\epsilon_0}{|\vec{x} - \vec{y}|} + \frac{q'/4\pi\epsilon_0}{|\vec{x} - \vec{y}'|}$$

$$q' = -\frac{aq}{y}; \quad \vec{y}' = \frac{a^2}{y^2} \vec{y}$$

The Dirichlet Green function is  $\bar{\Phi}_{\vec{y}}(\vec{x})$  for  $q = 4\pi\epsilon_0$

$$G_D(\vec{x}, \vec{y}) = \frac{1}{|\vec{x} - \vec{y}|} - \frac{a}{y |\vec{x} - \frac{a^2}{y^2} \vec{y}|}$$

Note: thanks to the general property

$G_D(\vec{x}, \vec{y}) = G_D(\vec{y}, \vec{x})$  the order of arguments is irrelevant.

$$|\vec{x} - \frac{a^2}{y^2} \vec{y}| = \frac{a}{y} |\vec{x} \frac{y}{a} - \vec{y}| = \frac{a}{y} \left( \frac{x^2 y^2}{a^2} + a^2 - 2xy \cos \delta \right)^{1/2}$$

where  $\delta$  is the angle between  $\vec{x}$  and  $\vec{y}$

$$\Rightarrow G_D(\vec{x}, \vec{y}) = \frac{1}{(x^2 + y^2 - 2xy \cos \delta)^{1/2}} - \frac{1}{\left( \frac{x^2 y^2}{a^2} + a^2 - 2xy \cos \delta \right)^{1/2}}$$

To comply with J.D.J. replace  $\vec{y} \rightarrow \vec{x}'$ .

To use our formalism we need the normal derivative  $\frac{\partial G_D(\vec{x}, \vec{x}')}{\partial n'}$  on a sphere



Note as the region of interest is outside the sphere, the normal  $\bar{n}'$  points inward!

It remains to use Eq. (4) with  $q = 4\pi\epsilon_0$  and replacing  $y \rightarrow x$  to obtain

$$\left. \frac{\partial G}{\partial n'} \right|_{x'=a} = - \frac{(x^2 - a^2)}{a(x^2 + a^2 - 2ax \cos \delta)^{3/2}}$$

With our master formula

$$\Phi(\vec{x}) = \frac{1}{4\pi\epsilon_0} \int_V d^3x' \rho(\vec{x}') G_D(\vec{x}, \vec{x}') - \frac{1}{4\pi} \oint_S da' \Phi(\vec{x}') \frac{\partial G_D(\vec{x}, \vec{x}')}{\partial n'}$$

we obtain the general solution of the Dirichlet problem outside the sphere:

$$\Phi(\vec{x}) = \frac{1}{4\pi\epsilon_0} \int_{|\vec{x}'| > a} d^3x' \rho(\vec{x}') \left[ \frac{1}{(x^2 + x'^2 - 2xx' \cos \delta)^{1/2}} - \frac{1}{\left(\frac{x^2 x'^2}{a^2} + a^2 - 2xx' \cos \delta\right)^{1/2}} \right]$$

$$+ \frac{1}{4\pi} \int d\Omega' \Phi(a, \theta', \phi') \frac{a(x^2 - a^2)}{(x^2 + a^2 - 2ax \cos \delta)^{3/2}}$$

$$\cos \delta \equiv \hat{x} \cdot \hat{x}' ; \quad \hat{x} = (\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta)$$

$$\hat{x}' = (\sin \theta' \cos \phi', \sin \theta' \sin \phi', \cos \theta')$$

$$\Rightarrow \cos \delta = \cos \theta \cos \theta' + \sin \theta \sin \theta' (\sin \phi \sin \phi' + \cos \phi \cos \phi') =$$

$$= \cos \theta \cos \theta' + \sin \theta \sin \theta' \cos(\phi - \phi')$$