

# ①

## Chapter 1.8 J.D.J. Green's Theorem

In generic electrostatics problem apart from the prescribed charge distribution  $\rho(\vec{x})$  there are surfaces with prescribed potentials or fields in many situations it is the potential (or field) that is prescribed rather than the charge density. In all such cases the explicit expression

$$\phi(\vec{x}) = \frac{1}{4\pi\epsilon_0} \int d^3x' \frac{\rho(\vec{x}')}{|\vec{x} - \vec{x}'|}$$

might not be directly applicable.

Goal: develop tools to treat the problems with specified Boundary Conditions.

Ist Green Identity:

$$\int_V d^3x (\phi \nabla^2 \psi + \bar{\nabla}\phi \cdot \bar{\nabla}\psi) = \oint_S d\sigma \phi \frac{\partial \psi}{\partial n}$$

where 1)  $S$  is the surface enclosing the volume  $V$ , 2)  $\phi, \psi$  are well behaved functions in  $V$ , 3)  $\frac{\partial \psi}{\partial n} = \bar{\nabla}\psi \cdot \vec{n}$ , i.e.  $\frac{\partial \psi}{\partial n}$  is the normal derivative with  $\vec{n}$  being a unit vector pointing outward from inside  $S$



By definition  $|\vec{n}|=1$

(2)

□ Start with Gauss Theorem

$$\int_V d^3x \nabla \cdot \vec{A} = \oint_S da \vec{A} \cdot \vec{n}$$

(\*)

with the choice  $\vec{A} = \phi \vec{\nabla} \psi$ 

$$\nabla \cdot (\phi \vec{\nabla} \psi) = \phi \nabla^2 \psi + \vec{\nabla} \phi \cdot \vec{\nabla} \psi$$

indeed  $[\nabla \cdot (\phi \vec{\nabla} \psi)] = \sum_{i=1}^3 \partial_i (\phi \partial_i \psi) =$

$$= \sum_{i=1}^3 \partial_i \phi \partial_i \psi + \sum_{i=1}^3 \phi \partial_i^2 \psi = \vec{\nabla} \phi \cdot \vec{\nabla} \psi + \phi \nabla^2 \psi$$

$\phi \vec{\nabla} \psi \cdot \vec{n} = \phi \frac{\partial \psi}{\partial n}$ ; substituting to the (\*)

$$\int_V d^3x (\phi \nabla^2 \psi + \vec{\nabla} \phi \cdot \vec{\nabla} \psi) = \oint_S da \phi \frac{\partial \psi}{\partial n}$$

(\*\*) (⊗)

2-nd Green's identity (Green's Theorem) reads  
THE STATEMENT

$$\int_V d^3x (\phi \nabla^2 \psi - \psi \nabla^2 \phi) = \oint_S da \left[ \phi \frac{\partial \psi}{\partial n} - \psi \frac{\partial \phi}{\partial n} \right]$$

(★)

□ exchange  $\phi \leftrightarrow \psi$  in (\*\*)

$$\int_V d^3x (\psi \nabla^2 \phi + \vec{\nabla} \psi \cdot \vec{\nabla} \phi) = \oint_S da \psi \frac{\partial \phi}{\partial n}$$

(V)

$$(V) - (**) \Rightarrow \int_V d^3x (\phi \nabla^2 \psi - \psi \nabla^2 \phi) =$$

$$= \oint_S da \left[ \phi \frac{\partial \psi}{\partial n} - \psi \frac{\partial \phi}{\partial n} \right] \quad \text{⊗}$$

## Integral constrain on solutions of Poisson Eq.

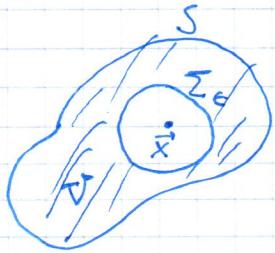
(3)

Let  $\Phi$  be a solution of  $\nabla^2 \Phi = -\frac{f}{\rho_0}$

Consider closed volume  $V$

enclosed by the surface  $S$

and take point  $\vec{x}$  inside  $V$



Take in Green's Theorem,  $\star$

$$\phi = \Phi, \quad \Psi_{\vec{x}}(\vec{x}') = \frac{1}{|\vec{x} - \vec{x}'|}$$

Note that in  $\Psi_{\vec{x}}(\vec{x}')$ ,  $\vec{x}$  is parameter and  $\vec{x}'$  is a variable.

As  $\Psi_{\vec{x}}(\vec{x}')$  is singular for  $\vec{x}' \rightarrow \vec{x}$  consider an alternative volume

$V' = V - B_x^\epsilon$ , where  $B_x^\epsilon$  is a ball of radius  $\epsilon$  centered at  $\vec{x}$ .

Eventually  $\epsilon$  will go to zero

For the volume  $V'$  the surface  $S' = S + \Sigma_\epsilon$

where  $\Sigma_\epsilon$  is the surface of the ball  $B_x^\epsilon$

Green's Theorem for  $V'$  takes the form

$$\int_{V'} d^3x' \left[ \Phi(\vec{x}') \nabla^2 \Psi_{\vec{x}}(\vec{x}') - \Psi_{\vec{x}}(\vec{x}') \nabla^2 \Phi(\vec{x}') \right] = \text{(Eq. 1)}$$

$$= \oint_S d\alpha' \left( \Phi \frac{\partial \Psi}{\partial n} - \Psi \frac{\partial \Phi}{\partial n} \right) + \oint_{\Sigma_\epsilon} d\alpha' \left( \Phi \frac{\partial \Psi}{\partial n} - \Psi \frac{\partial \Phi}{\partial n} \right)$$

(4)

1) inside  $V'$ ,  $\bar{x} \neq \bar{x}' \Rightarrow \nabla_{\bar{x}'}^2 \Psi_{\bar{x}}(\bar{x}') = 0$

$$\nabla^2 \Phi = -\frac{1}{\epsilon_0} \rho$$

$\Rightarrow$  LHS of Eq.(1) becomes

$$\int_{V'} d^3x' \frac{1}{\epsilon_0} \frac{\rho(\bar{x}')}{|\bar{x} - \bar{x}'|} = \int_{V - B_{\bar{x}}^{\epsilon}} d^3x' \frac{1}{\epsilon_0} \frac{\rho(\bar{x}')}{|\bar{x} - \bar{x}'|}$$

Take the limit  $\epsilon \rightarrow 0$

$$\lim_{\epsilon \rightarrow 0} \text{LHS} = \int_V d^3x' \frac{1}{\epsilon_0} \frac{\rho(\bar{x}')}{|\bar{x} - \bar{x}'|}$$

Consider the RHS of Eq. 1

only the integral over  $\Sigma_{\epsilon}$  depends on  $\epsilon$

$$1) \left| \oint_{\Sigma_{\epsilon}} da' \Psi \frac{\partial \Phi}{\partial n} \right| \leq 4\pi \epsilon^2 \frac{1}{\epsilon} \max_{\bar{x}' \in \Sigma_{\epsilon}} \left| \frac{\partial \Phi}{\partial n} \right| \xrightarrow{\epsilon \rightarrow 0} 0$$

$$2) \oint_{\Sigma_{\epsilon}} da' \Psi \frac{\partial \Phi}{\partial n} = \frac{1}{\epsilon^2} \oint_{\Sigma_{\epsilon}} da' \Phi = \frac{1}{\epsilon^2} 4\pi \epsilon^2 \Phi(\bar{x}) \quad \text{for some } \bar{x} \in \Sigma_{\epsilon}$$

$$\Rightarrow \oint_{\Sigma_{\epsilon}} da' \Phi \frac{\partial \Psi}{\partial n} \xrightarrow{\epsilon \rightarrow 0} 4\pi \Phi(\bar{x})$$

since  $\Phi(\bar{x})$  is continuous

Combining all pieces together and dividing by  $(4\pi)$

$$\Phi(\bar{x}) = \frac{1}{4\pi \epsilon_0} \int_V d^3x' \frac{\rho(\bar{x}')}{R} + \frac{1}{4\pi} \oint_S da' \left[ \frac{1}{R} \frac{\partial \Phi}{\partial n'} - \Phi \frac{\partial}{\partial n'} \left( \frac{1}{R} \right) \right] \quad (\text{Eq. 2})$$

where  $\frac{1}{R} \equiv \frac{1}{|\bar{x} - \bar{x}'|} \equiv \Psi_{\bar{x}}(\bar{x}')$  holds for  $\bar{x}$  inside  $V$

(5)

If on the other hand  $\vec{x}$  is outside the volume  $V$  the derivation is the same but no  $\oint_{\Sigma_e}$  integral as a result

for  $\vec{x}$  outside  $V$

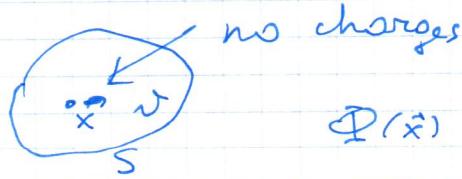
$$O = \frac{1}{4\pi\epsilon_0} \int_V d^3x' \frac{\rho(\vec{x}')}{R} + \frac{1}{4\pi} \oint_S d\sigma' \left[ \frac{1}{R} \frac{\partial \Phi}{\partial n'} - \Phi \frac{\partial}{\partial n'} \left( \frac{1}{R} \right) \right] \quad (\text{Eq. 3})$$

### Remarks

- 1) IF the boundary condition is that the potential falls to zero at infinity we are back to

$$\Phi(\vec{x}) = \frac{1}{4\pi\epsilon_0} \int d^3x' \frac{\rho(\vec{x}')}{R}$$

- 2) If there are no charges in  $V$

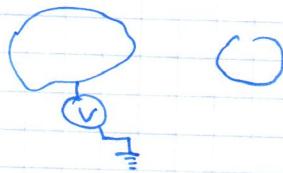


$$\Phi(\vec{x}) = \frac{1}{4\pi} \oint_S d\sigma' \left[ \frac{1}{R} \frac{\partial \Phi}{\partial n'} - \Phi \frac{\partial}{\partial n'} \left( \frac{1}{R} \right) \right]$$

this is a constrain that any solution has to satisfy. But its not a solution since  $\Phi$  and  $\frac{\partial \Phi}{\partial n'}$  cannot be specified independently on a surface  $S$ .

## Uniqueness of the Solution with Dirichlet or Neumann Boundary Conditions (BC)

Dirichlet BC : specify the potentials on a closed surfaces



(conductors under the given voltage)

Neumann B.C. : specify electric field (normal derivatives) of the potential on surfaces

The solution of Poisson equation is unique in both cases.

□

$$\nabla^2 \Phi = -\rho/\epsilon_0 \text{ inside a volume } V$$

Let's say we have two solutions  $\Phi_1, \Phi_2$  satisfying the same BC.

$$\text{Define } U = \Phi_2 - \Phi_1$$

$$\Rightarrow \nabla^2 U = \nabla^2 \Phi_2 - \nabla^2 \Phi_1 = -\rho/\epsilon_0 + \rho/\epsilon_0 = 0 \text{ in } V$$

$$\text{Dirichlet} \Rightarrow U = 0 \text{ on } S; \text{ Neumann} \Rightarrow \frac{\partial U}{\partial n} = 0 \text{ on } S$$

From (\*\*\*) p.2 [Green's first identity]

with  $\phi = \psi = U$  one gets

$$\int_V d^3x \left( U \nabla^2 U + (\nabla U)^2 \right) = \oint_S d\alpha \underset{\substack{\circ \\ \text{Dirichlet}}}{U} \underset{\substack{\circ \\ \text{Neumann}}}{\frac{\partial U}{\partial n}} = 0$$

$$\Rightarrow \int_V d^3x (\nabla U)^2 = 0 \Rightarrow \nabla U \equiv 0 \text{ in } V \Rightarrow U = \text{const in } V$$

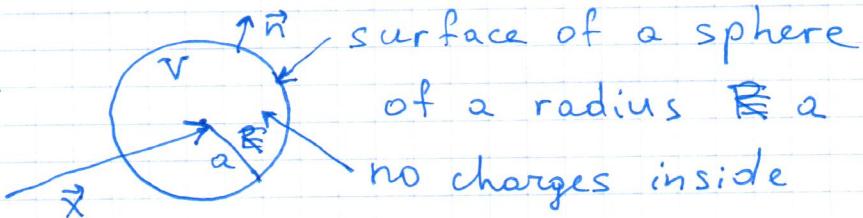
(7)

Dirichlet  $\Rightarrow U = \text{const} \Rightarrow \nabla \Phi = 0 \Rightarrow \Phi_1 = \Phi_2$   
 Neumann  $\Rightarrow U = \text{const} \Rightarrow \Phi_1 - \Phi_2 = \text{const}$   
 As the potential is defined up to a constant  
 anyway the solution is unique  $\square$

Chapter Example Problem 1.10 J.D.J.

Prove the mean value theorem: For charge free space the value of the electrostatic potential at any point is equal to the average of the potential over the surface of any sphere centered on that point.

Solution:  $\square$



(Eq.2) p.4  $\Rightarrow$

$$\Phi(\vec{r}) = \frac{1}{4\pi\epsilon_0} \int_V d^3x' \frac{\rho(\vec{x}')}{R} + \frac{1}{4\pi} \oint_S ds' \left[ \frac{1}{R} \frac{\partial \Phi}{\partial n'} - \Phi \frac{\partial}{\partial n'} \left( \frac{1}{R} \right) \right]$$

1-st term vanishes as  $\rho(\vec{x}') = 0$  in  $V$

Recall :  $R = |\vec{x}' - \vec{x}|$

$R \equiv a$  on  $S$

Gauss

$$\frac{\partial \Phi}{\partial n'} = -\vec{E} \cdot \vec{n} \Rightarrow \oint_S ds' \frac{1}{R} \frac{\partial \Phi}{\partial n'} = \frac{1}{R} \oint_S ds' (-\vec{E} \cdot \vec{n})$$

Total charge in  $V$ ,  $Q = 0$

$$= -\frac{1}{R} \frac{1}{\epsilon_0} Q \stackrel{Q=0}{=} 0 \Rightarrow \text{2-nd term vanishes}$$

3-rd term is the only term that remains

(8)

$$\Phi(\vec{x}) = -\frac{1}{4\pi} \oint_S ds' \Phi \frac{\partial}{\partial n'} \left( \frac{1}{R} \right)$$

But  $\frac{\partial}{\partial n'} \left( \frac{1}{R} \right) = -\frac{1}{R^2} \Rightarrow$

$$\begin{aligned} \Rightarrow \Phi(\vec{x}) &= \frac{1}{4\pi} \oint_S ds' \frac{1}{R^2} \Phi(\vec{x}') \stackrel{R \equiv 0 \text{ on } S}{=} \frac{1}{4\pi R^2} \oint_S ds' \Phi(\vec{x}') = \\ &= \frac{\oint_S ds' \Phi(\vec{x}')}{\oint_S ds'} = \langle \Phi \rangle_S \quad \blacksquare \end{aligned}$$

### Chapter 1.10

Let's start with the generalization of our derivation of (Eq. 2) p. 4.

Observe that instead of  $\frac{1}{R} = \frac{1}{|\vec{x} - \vec{x}'|} = \Psi_{\vec{x}}(\vec{x}')$

one could equally use

$$G(\vec{x}, \vec{x}') = \frac{1}{|\vec{x} - \vec{x}'|} + F(\vec{x}, \vec{x}')$$

for any  $F(\vec{x}, \vec{x}')$  satisfying  $\nabla_{\vec{x}'}^2 F(\vec{x}, \vec{x}') = 0$ .

Indeed:  $\nabla_{\vec{x}'}^2 G(\vec{x}, \vec{x}') = 0$  for  $\vec{x} \neq \vec{x}'$

and  $G(\vec{x}, \vec{x}') \rightarrow \frac{1}{|\vec{x} - \vec{x}'|}$  as  $\vec{x} \rightarrow \vec{x}'$  and  $F(\vec{x}, \vec{x}')$

is regular.

As a result we have for  $\vec{x}$  inside  $V$

$$\Phi(\vec{x}) = \frac{1}{4\pi \epsilon_0} \int_V d^3x' g(\vec{x}') G(\vec{x}, \vec{x}')$$

$$+ \frac{1}{4\pi} \oint_S ds' \left[ G(\vec{x}, \vec{x}') \frac{\partial \Phi}{\partial n'} - \Phi(\vec{x}') \frac{\partial G(\vec{x}, \vec{x}')}{\partial n'} \right] \quad (\text{Eq. 5})$$

Let's use the freedom to chose  $F(\vec{x}, \vec{x}')$  to  
solve Dirichlet BC problem

Let's say we have found

$$G_D(\vec{x}, \vec{x}') = 0 \text{ for } \vec{x}' \text{ on } S$$

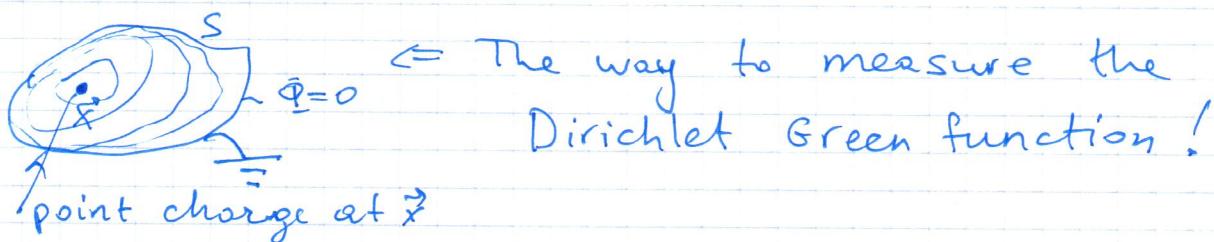
then for this choice of  $G(\vec{x}, \vec{x}')$  we have  
from Eq. 5)

$$\Phi(\vec{x}) = \frac{1}{4\pi\epsilon_0} \int_V d^3x' \rho(\vec{x}') G_D(\vec{x}, \vec{x}') - \frac{1}{4\pi} \oint_S ds' \Phi(\vec{x}') \frac{\partial G_D}{\partial n'} \quad (\text{Eq. 6})$$

Once  $G_D(\vec{x}, \vec{x}')$  is known  $\Rightarrow$  the solution of the  
Dirichlet BC problem is given by (Eq. 6)

Note  $G_D(\vec{x}, \vec{x}')$  knows only about the shape  
of the surfaces.

$G_D(\vec{x}, \vec{x}')$  in the case of one closed surface  
is the potential of a <sup>point</sup> unit charge at  $\vec{x}$   
when the surface  $S$  is grounded:



Neumann BC. one could try to find

$$\frac{\partial G_N(\vec{x}, \vec{x}')}{\partial n'} = 0 \text{ for } \vec{x}' \text{ on } S$$

But this is impossible: By Gauss law

$$\oint_S ds' \frac{\partial G}{\partial n'} = \oint_S ds' \vec{\nabla}_{\vec{x}'} \left[ \frac{1}{|\vec{x} - \vec{x}'|} + F(\vec{x}, \vec{x}') \right] \cdot \vec{n} =$$

$$= \oint_S dS' \hat{n} \cdot \vec{\nabla}_{\bar{x}'} \frac{1}{|\bar{x} - \bar{x}'|} + \oint_S dS' [\vec{\nabla}_{\bar{x}'} F(\bar{x}, \bar{x}')] \cdot \hat{n}$$

The second term gives by the divergence theorem

$$\oint_S dS' [\vec{\nabla}_{\bar{x}'} F(\bar{x}, \bar{x}')] \cdot \hat{n} = \int_V d^3x' \underbrace{\nabla_{\bar{x}'}^2 F(\bar{x}, \bar{x}')}_{\delta''} = 0$$

In the first term  $\frac{1}{|\bar{x} - \bar{x}'|}$  is the potential of point

$eV$  charge  $4\pi\epsilon_0$ ,  $\vec{\nabla}_{\bar{x}'} \frac{1}{|\bar{x} - \bar{x}'|}$  is the field

this charge (at  $\bar{x}'$ ) produces at  $\bar{x}'$

$$\Rightarrow \oint_S dS' \hat{n} \cdot \vec{\nabla}_{\bar{x}'} \frac{1}{|\bar{x} - \bar{x}'|} = -\frac{4\pi\epsilon_0}{\epsilon_0} = -4\pi$$

$$\Rightarrow \oint_S dS' \frac{\partial G}{\partial n'} = -4\pi$$

$$\text{One could try } \frac{\partial G_N}{\partial n'} (\bar{x}, \bar{x}') = -\frac{4\pi}{S}, \bar{x}' \text{ on } S$$

where  $S$  is the total area of ~~the~~ surfaces

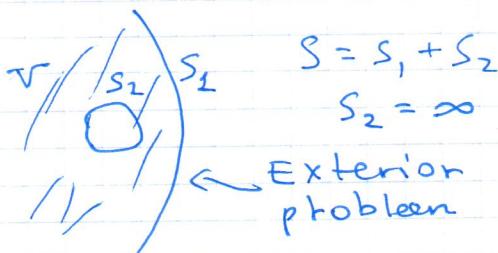
bounding the volume of interest  $V$

with this choice (Eq. 6) gives

$$\Phi(\bar{x}) = \langle \Phi \rangle_S + \frac{1}{4\pi\epsilon_0} \int_V \rho(\bar{x}') G_N(\bar{x}, \bar{x}') d^3x' + \frac{1}{4\pi} \oint_S dS' \frac{\partial \Phi}{\partial n'} G_N$$

We will be interested in exterior Neumann problem

when the total area  $S = \infty$



In the case of exterior Neumann problem (11)

$$\frac{\partial G_N(\bar{x}, \bar{x}')}{\partial n'} = -\frac{4\pi}{S} \rightarrow 0 \quad \text{for } \bar{x}' \text{ on } S$$

and  $\langle \Phi \rangle_s \Rightarrow 0$ ; as a result

$$\Phi(\bar{x}) = \frac{1}{4\pi \epsilon_0} \int_V d^3x' \rho(\bar{x}') G_N(\bar{x}, \bar{x}') + \frac{1}{4\pi} \int_S ds' \frac{\partial \Phi}{\partial n'} G_N(\bar{x}, \bar{x}')$$

is the general solution of exterior Neumann problem by the Neumann Green function satisfying

$$\frac{\partial G_N(\bar{x}, \bar{x}')}{\partial n'} = 0 \quad \text{for } \bar{x}' \text{ on } S$$