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# Ch. 13 Magnetic Matter

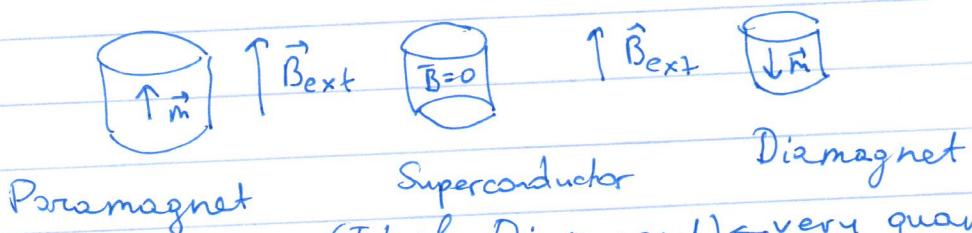
## 13.1 Intro

$$\vec{B}_{\text{tot}}(\vec{r}) = \vec{B}_{\text{ext}}(\vec{r}) + \vec{B}_{\text{self}}(\vec{r})$$

↑ external magnetic field  
↑ the field produced by the matter in response

Purely Quantum Phenomenon

At large distance from the system (matter) the magnetic field is dipolar, and is characterized by a macroscopic magnetic moment  $\vec{m}$



Paramagnet

Superconductor

Diamagnet

(Ideal Diamagnet) ← very quantum

For most magnets  $\vec{m} \rightarrow 0$  as  $B_{\text{ext}} \rightarrow 0$

exception: Ferromagnets (subclass of paramagnets)

with frozen dipole moment  $\vec{m}$

Permanent magnets: NdFeB, SmCo, AlNiCo

## 13.2 Magnetization

Let  $\vec{B}_{\text{ext}}(\vec{r})$  be caused by the current distribution  $\vec{j}(\vec{r})$ .  
(it is commonly referred to as free current)

$$\vec{B}_{\text{ext}} \longrightarrow \uparrow \quad \vec{B}_{\text{self}} = \vec{B}_M$$

$$\vec{j}(\vec{r}) = \vec{j}_f(\vec{r}) + \vec{j}_m(\vec{r})$$

### 13.2.1

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#### Spin Magnetization

Collection of  $N$  electrons with spin magnetic moments

$\vec{m}_k$  (all with the same magnitude) gives spin magnetization

$$\vec{M}_S(\vec{r}) = \sum_{k=1}^N \vec{m}_k \delta(\vec{r} - \vec{r}_k)$$

Magnetic field produced by a point magnetic moment  $\vec{m}$  at the origin is identical to the current

$$\vec{j} = \vec{\nabla} \times [\vec{m} \delta(\vec{r})] \Rightarrow$$

The magnetic field produced by  $\vec{M}_S(\vec{r})$  is the same as produced by spin magnetization current density

$$\vec{j}_S(\vec{r}) \stackrel{\rightarrow}{\nabla} \times \vec{M}_S(\vec{r})$$

Typically spins are confined to a volume of a magnet

$$\vec{j}_S(\vec{r}) = \vec{\nabla} \times [\vec{M}_S \Theta(-z)] = [\vec{\nabla} \times \vec{M}_S] \Theta(-z) + [\vec{M}_S \times \vec{z}] \delta(z)$$

Bulk spin magnetization current

spin magnetization surface current

$$\vec{R}_S(x, y) = \int dz \vec{j}_S = \vec{M}_S(x, y) \times \hat{z}$$

in general  $\vec{R}_S(\vec{r}_S) = \vec{M}_S(\vec{r}_S) \times \hat{n}(\vec{r}_S)$

### 13.2.2 Orbital magnetization

$\vec{j}_O$  orbital magnetization current density can be thought of representing current loops circulating in the body.

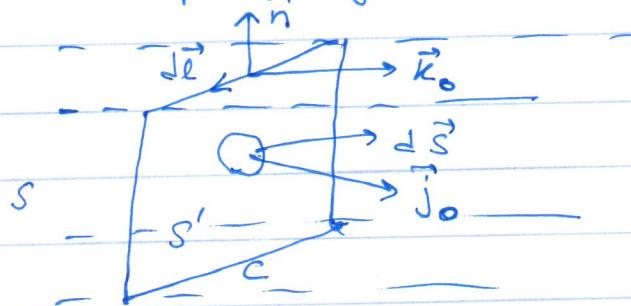
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The key observation :  
no net current  $I$  through any cross-sectional  
surface  $S'$  of a finite sample



If that would not be so  
the equilibrium would not be possible

Let's quantify this idea



$$I = \int_{S'} d\vec{S} \cdot \vec{j}_0(\vec{r}) +$$

$$+ \oint_C d\vec{l} \cdot \vec{k}_0(\vec{r}) \times \hat{n}(\vec{r}_S) = 0$$

$\vec{k}_0(\vec{r}) \cdot (\hat{n} \times d\vec{l})$

What functions  $\vec{j}_0(\vec{r})$ ,  $\vec{k}_0(\vec{r})$  satisfy the  
above condition for all  $C, S'$ ?  
Write the answer in terms of  $\vec{M}_0(\vec{r})$ :

$$\vec{j}_0(\vec{r}) = \vec{\nabla} \times \vec{M}_0(\vec{r}), \vec{r} \in V$$

$$\vec{k}_0(\vec{r}_S) = \vec{M}_0(\vec{r}_S) \times \hat{n}(\vec{r}_S), \vec{r}_S \in S, \boxed{\vec{M}_0(\vec{r}) = 0, \vec{r} \notin V}$$

Let's check it :

$$\int_{S'} d\vec{S} \cdot \vec{j}_0 = \int_{S'} d\vec{S} \cdot (\vec{\nabla} \times \vec{M}_0) = \oint_C d\vec{l} \cdot \vec{M}_0$$

$$\oint_C d\vec{l} \cdot \vec{k}_0(\vec{r}_S) \times \hat{n}(\vec{r}_S) = \oint_C d\vec{l} \cdot (\vec{M}_0 \times \hat{n}(\vec{r})) \times \hat{n}(\vec{r}_S) =$$

$$= \oint_C d\vec{l} (-\vec{M}_0 + \hat{n}(\vec{r}_S)(\vec{M}_0 \cdot \hat{n}(\vec{r}_S))) = - \oint_C \vec{M}_0 \cdot d\vec{l}$$

$\hat{n}(\vec{r}_S) \perp d\vec{l}$   
gives zeros

⊗ Do not fix  $\vec{M}_0$  ( $\vec{\nabla} \cdot \vec{M}_0$  is not given)

### 13.2.3 Total Magnetization

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$$\vec{M}(\vec{r}) = \vec{M}_S(\vec{r}) + \vec{M}_B(\vec{r}) \Rightarrow$$

$$\vec{j}_M(\vec{r}) = \vec{\nabla} \times \vec{M}(\vec{r}), \quad \vec{k}_M(\vec{r}) = \vec{M}(\vec{r}) \times \hat{n}(\vec{r}) \quad (A)$$

### 13.2.4 The volume integral of $\vec{M}(\vec{r})$

For the spin part

$$\vec{m}_S = \int d^3r \sum_{k=1}^n \vec{m}_k \delta(\vec{r} - \vec{r}_k) = \sum_{k=1}^n \vec{m}_k \equiv \vec{M}_S$$

$\uparrow \quad \vec{M}_S(\vec{r})$

spin magnetic moment.

Consider the orbital part

$$\vec{m} = \frac{1}{2} \int_V d^3r [\vec{r} \times \vec{j}_M] + \frac{1}{2} \int_S d\vec{s} [\vec{r} \times \vec{k}_M]$$

Substitute in this equation (A) to get

$$[\vec{r} \times \vec{j}_M]_k = [\vec{r} \times (\vec{\nabla} \times \vec{M}(\vec{r}))]_k = \epsilon_{kem} r_e (\vec{\nabla} \times \vec{M})_m =$$

$$= \epsilon_{kem} r_e \epsilon_{mst} \partial_s M_t = (\delta_{ks} \delta_{et} - \delta_{kt} \delta_{es}) r_e \partial_s M_t =$$

$$= r_e \partial_k M_e - r_e \partial_e M_k$$

$$[\vec{r} \times \vec{k}_M]_k = [\vec{r} \times (\vec{M} \times \hat{n})]_k = M_k (\vec{r} \cdot \hat{n}) - \hat{n}_k (\vec{M} \cdot \vec{r})$$

original expression

$$m_k = \frac{1}{2} \int_V d^3r [ \underbrace{\partial_k(r_e M_e)}_{cancel} - M_e \partial_e r_e ] - \frac{1}{2} \int_V d^3r [ \underbrace{\partial_e(r_e M_k)}_{cancel} - M_k \partial_e r_e ]$$

$$+ \frac{1}{2} \int_S d\vec{s} [ \hat{n}_e M_k r_e - \hat{n}_k r_e M_e ] \quad \text{cancel}$$

$$= \frac{1}{2} \int_V d^3r [ \underbrace{-M_e \partial_k r_e}_{\delta_{k,e}} + M_k \underbrace{\partial_e r_e}_{''3} ] = \int_V d^3r M_k \quad \boxed{4}$$

### 13.2.5 The Lorentz Model

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$$\vec{M}(\vec{r}) = \frac{1}{2\pi} \int_S d^3s \vec{s} \times \vec{j}_{m, \text{micro}}(\vec{s}) = \frac{\vec{m}(\vec{r})}{S}$$

→ microscopic cell

true for spin part, but is approximately true for magnetic insulator, and may be not true for a conductor: currents flow in and out of the cell.

### 13.2.6 The Non-Uniqueness of $\vec{M}(\vec{r})$

$\vec{j}_o = \vec{\nabla} \times \vec{M}_o$  cannot be uniquely inverted

$$\vec{M}_o \rightarrow \vec{M}_o + \vec{\nabla} \Lambda, \quad \Lambda \text{ must be constant on } S$$

$$(E = \vec{M} \times \vec{n}, \vec{\nabla} \Lambda \times \vec{n} = 0 \text{ if } \Lambda \text{ changes in direction } \parallel \vec{n})$$

### 13.3 The Field Produced by Magnetized Matter

$$\vec{A}_m = \frac{\mu_0}{4\pi} \int_V d^3r' \frac{\vec{\nabla}' \times \vec{M}(\vec{r}')}{|\vec{r} - \vec{r}'|} + \frac{\mu_0}{4\pi} \int_S ds' \frac{\vec{M}(\vec{r}') \times \vec{n}'}{|\vec{r} - \vec{r}'|}$$

$$(\vec{\nabla}_n (\vec{E} \times \vec{A}) = \mu_0 j')$$

$$(\vec{\nabla} \cdot \vec{A} = 0, -\nabla^2 \Phi \mu_0)$$

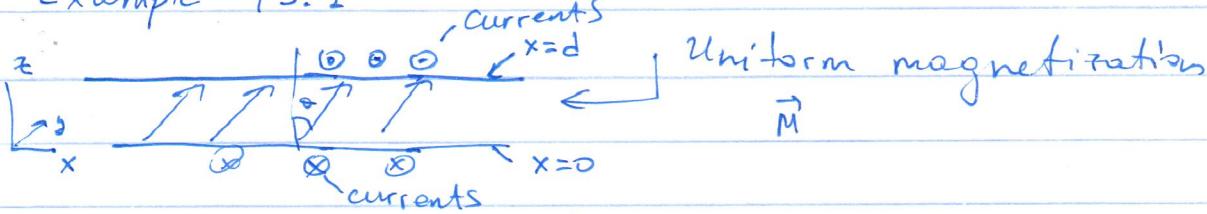
Biot-Savart:

$$\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi} \int_V d^3r' \frac{[\vec{\nabla}' \times \vec{M}(\vec{r}')] \times (\vec{r} - \vec{r}')}{|\vec{r} - \vec{r}'|^3} + \frac{\mu_0}{4\pi} \int_S ds' \frac{[\vec{M}(\vec{r}') \times \vec{n}'] \times (\vec{r} - \vec{r}')}{|\vec{r} - \vec{r}'|^3}$$

In many cases it is not necessary to evaluate these integrals explicitly

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## Example 13.1



$\nabla \times \vec{M} = 0 \Rightarrow$  no volume current density

$$\vec{B}_M = \vec{M} \times \hat{n}, (\vec{B}_M) = M \sin \theta$$

$$\hat{n} = \begin{cases} \hat{x} & x=d \\ -\hat{x} & x=0 \end{cases} \Rightarrow \vec{M} \times \hat{n} = \begin{cases} -M \hat{y} \sin \theta & x=d \\ +M \hat{y} \sin \theta & x=0 \end{cases}$$

$$\Rightarrow \vec{B}_M (\vec{r}) = \begin{cases} M \sin \theta \hat{x} & \text{inside the slab} \\ 0 & \text{outside} \end{cases}$$

13.3.1 Magnetized Matter as a superposition of Point Dipoles  
Write the surface integral of  $\vec{A}_m$  on page 5 as

$$\oint_S dS' \frac{\vec{M}(\vec{r}') \times \hat{n}'}{|\vec{r} - \vec{r}'|} = - \int_V d^3 r' \vec{\nabla}' \times \frac{\vec{M}(\vec{r}')}{|\vec{r} - \vec{r}'|}$$

$$\square \left[ \vec{\nabla}' \times \frac{\vec{M}(\vec{r}')}{|\vec{r} - \vec{r}'|} \right]_e = \epsilon_{emn} \frac{\partial}{\partial r'_m} \frac{M_n(\vec{r}')}{|\vec{r} - \vec{r}'|}$$

$$\Rightarrow \int_V d^3 r' \left[ \vec{\nabla}' \times \frac{\vec{M}(\vec{r}')}{|\vec{r} - \vec{r}'|} \right]_e = \epsilon_{emn} \oint_S dS' \hat{n}_m \frac{M_n(\vec{r}')}{|\vec{r} - \vec{r}'|} \quad \boxed{13}$$

$$\left[ \vec{\nabla}' \times \frac{\vec{M}(\vec{r}')}{|\vec{r} - \vec{r}'|} \right]_e = \epsilon_{emn} \frac{\partial}{\partial r'_m} \frac{M_n(\vec{r}')}{|\vec{r} - \vec{r}'|} = \left[ \frac{\vec{\nabla}' \times \vec{M}(\vec{r}')}{|\vec{r} - \vec{r}'|} \right]_e +$$

$$+ \left[ \left( \vec{\nabla}' \frac{1}{|\vec{r} - \vec{r}'|} \right) \times \vec{M}(\vec{r}') \right]_e$$

$$\oint_S dS' \frac{\vec{M}(\vec{r}') \times \hat{n}'}{|\vec{r} - \vec{r}'|} = - \int_V d^3 r' \frac{\vec{\nabla}' \times \vec{M}(\vec{r}')}{|\vec{r} - \vec{r}'|} + \int_V d^3 r' \vec{M}(\vec{r}') \times \vec{\nabla}' \frac{1}{|\vec{r} - \vec{r}'|}$$

↑ cancels with the volume contribution to  $\vec{A}_m$  on page 5

$$\vec{\nabla}' |\vec{r} - \vec{r}'|^3 = -\vec{\nabla} |\vec{r} - \vec{r}'|^3 = \frac{\vec{r} - \vec{r}'}{|\vec{r} - \vec{r}'|^3}$$

(7)

$$\vec{A}_m(\vec{r}) = \frac{\mu_0}{4\pi} \int_V d^3 r' \vec{M}(\vec{r}') \times \frac{\vec{r} - \vec{r}'}{|\vec{r} - \vec{r}'|^3}$$

$\Rightarrow$  the vector potential of the dipolar distribution

$d^3 r' \vec{M}(\vec{r}') = d\vec{m}$  = The dipole moment  
of an elementary volume  $d^3 r'$

$\vec{M}(\vec{r})$  is the density of dipole moment at  
least for the purpose of computing  $\vec{A}_m(\vec{B}_m)$

$$(\vec{\nabla} \times \vec{M}) \cdot \hat{n} = \lim_{dS \rightarrow 0} \frac{1}{dS} \oint_C d\vec{l} \cdot \vec{M}$$



Let's develop an alternative to Biot-Savart

located at  $\vec{r}'$

$$\text{For point dipole } \checkmark \quad \vec{B}(\vec{r}) = \mu_0 \left[ \vec{m} \delta(\vec{r} - \vec{r}_0) - \vec{\nabla} \frac{1}{4\pi} \frac{\vec{m} \cdot (\vec{r} - \vec{r}')}{|\vec{r} - \vec{r}'|^3} \right]$$

$$\vec{B}_m(\vec{r}) = \mu_0 \int_V d^3 r' \vec{M}(\vec{r}') \delta(\vec{r} - \vec{r}') - \vec{\nabla} \frac{\mu_0}{4\pi} \int_V d^3 r' \vec{M}(\vec{r}') \cdot \frac{\vec{r} - \vec{r}'}{|\vec{r} - \vec{r}'|^3}$$

Define

$$\boxed{\Psi_m(\vec{r}) = \frac{1}{4\pi} \int_V d^3 r' \vec{M}(\vec{r}') \cdot \vec{\nabla}' \frac{1}{|\vec{r} - \vec{r}'|}} \quad \begin{array}{l} \text{magnetic} \\ \text{scalar} \\ \text{potential} \end{array}$$

$$\vec{B}_m(\vec{r}) = \begin{cases} \mu_0 \vec{M}(\vec{r}) - \mu_0 \vec{\nabla} \Psi_m(\vec{r}) & \vec{r} \in V \\ -\mu_0 \vec{\nabla} \Psi_m(\vec{r}) & \vec{r} \notin V \end{cases}$$

Define auxiliary field

$$\boxed{\vec{H}_m(\vec{r}) = -\vec{\nabla} \Psi_m(\vec{r})}$$

$$\vec{B}_m(\vec{r}) = \mu_0 [ \vec{M}(\vec{r}) + \vec{H}_m(\vec{r}) ] \rightarrow \text{holds for } \vec{r} \in V \text{ and } \vec{r} \notin V$$

## 13.4 Fictitious Magnetic Charge

(8)

Rewrite the definition of  $\chi_m(\vec{r})$  on page (7) as

$$\chi_m(\vec{r}) = \frac{1}{4\pi} \int_V d^3 r' \vec{\nabla}' \cdot \left[ \frac{\vec{M}(\vec{r}')}{|\vec{r} - \vec{r}'|} \right] - \frac{1}{4\pi} \int_V d^3 r' \frac{\vec{\nabla}' \cdot \vec{M}(\vec{r}')}{|\vec{r} - \vec{r}'|}$$

$$\begin{aligned} \square \quad \vec{\nabla} [\vec{f}(\vec{r}) g(\vec{r})] &= \frac{\partial}{\partial r_e} (f_e g) = \frac{\partial f_e}{\partial r_e} g + f_e \frac{\partial g}{\partial r_e} = \\ &= (\vec{\nabla} \cdot \vec{f}) \cdot g + \vec{f} \cdot \vec{\nabla} g \quad \square \end{aligned}$$

$$\chi_m(\vec{r}) = \frac{1}{4\pi} \int_V d^3 r' \frac{g^*(\vec{r}')} {|\vec{r} - \vec{r}'|} + \frac{1}{4\pi} \int_S dS \frac{\sigma^*(\vec{r}_S)} {|\vec{r} - \vec{r}_S|} \quad (*)$$

where  $\underbrace{g^*(\vec{r})}_{\substack{\downarrow \\ \text{volume}}} \equiv -\vec{\nabla} \cdot \vec{M}(\vec{r})$ ,  $\underbrace{\sigma^*(\vec{r}_S)}_{\substack{\downarrow \\ \text{surface}}} = \vec{M}(\vec{r}_S) \cdot \hat{n}(\vec{r}_S)$   
 density of fictitious magnetic charge.

$\vec{H}_m(\vec{r}) = -\vec{\nabla} \chi_m(\vec{r})$  is electric-like field.

$$\Rightarrow \text{Coulomb Law: } \vec{H}_m(\vec{r}) = \frac{1}{4\pi} \int_V d^3 r' g^*(\vec{r}') \frac{\vec{r} - \vec{r}'}{|\vec{r} - \vec{r}'|^3} + \frac{1}{4\pi} \int_S \sigma(\vec{r}_S) \frac{\vec{r} - \vec{r}_S}{|\vec{r} - \vec{r}_S|^3}$$

Back to the magnetized slab:

$$\begin{array}{c} \uparrow \quad \downarrow \\ \vec{H}_m = \vec{N} \cos \theta + \vec{M} \cos \phi = \vec{\sigma}^* \\ \downarrow \quad \uparrow \\ \vec{H}_m = \vec{N} \cos \theta - \vec{M} \cos \phi = \vec{\sigma}^* \end{array} \Rightarrow \vec{H}_m = \hat{z} (-\vec{\sigma}^*) \quad \begin{array}{l} \text{Like in electrostatics} \\ \text{but no } \epsilon_0! \end{array}$$

capacitor  $\vec{N}$   $\vec{H}_m$

$$\vec{B}_m(\vec{r}) = \begin{cases} \mu_0 ((\hat{x} M \cos \theta + \hat{y} M \sin \theta) + \hat{z} (-M \cos \theta)) = M \mu_0 \hat{x} \sin \theta \\ 0 + 0 = 0 \end{cases}$$

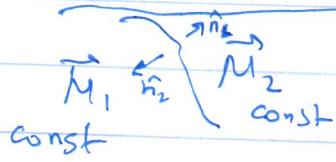
as before: description in terms of fictitious charges  
 is equivalent to that in terms of magnetization currents.

### 13.4.1 Potential Theory for $\vec{H}_m(\vec{r})$

(9)

$$\vec{\nabla} \times \vec{H}_m = 0, \quad \vec{\nabla} \cdot \vec{H}_m = -\vec{\nabla} \cdot \vec{M} = \rho^*, \quad \vec{H}_m = -\vec{\nabla} \psi_m$$

$$\Rightarrow \vec{\nabla}^2 \psi_m = \vec{\nabla} \cdot \vec{M} = -\rho^* \quad [\text{compose with } \vec{\nabla}^2 \vec{M} = -\rho \hat{e}_z]$$



Suppose we are in the region of a magnet with a constant  $\vec{M}$

$$\Rightarrow \vec{\nabla} \cdot \vec{M} = 0 = \rho^* = 0 \Rightarrow \vec{\nabla}^2 \psi_m = 0$$

Follows from explicit expression (8) on page (8)

$\psi$  is continuous  $\Rightarrow \psi_1(\vec{r}_S) = \psi_2(\vec{r}_S)$

Boundary conditions

$$\left( \frac{\partial \psi_1}{\partial n_1} - \frac{\partial \psi_2}{\partial n_1} \right) = \sigma_1^* + \sigma_2^* = [\vec{M}_1 - \vec{M}_2] \cdot \hat{n}_1$$

$\Rightarrow$  these equations uniquely specify  $\vec{B}_m(\vec{r})$  and  $\vec{H}_m(\vec{r})$

### Application 13.1 A uniformly magnetized sphere

$$\vec{M}(\vec{r}) = M \hat{z}$$

$$\vec{\sigma}^* = \vec{M} \cdot \hat{n}$$

$$\frac{\partial \psi_{in}}{\partial r} - \frac{\partial \psi_{out}}{\partial r} = \sigma^* = M \cos \theta$$

$$\vec{k}_S = \vec{M} \times \hat{n}$$

$$\psi_{in} = A \cos \theta, \quad \psi_{out} = B \cos \theta / r^2$$

$$\text{Continuity} \quad AR = B/R^2 \rightarrow \frac{2B}{R^3} + A = M \Rightarrow$$

$$\Rightarrow A = B/R^3 \Rightarrow 3B/R^3 = M, \quad B = \frac{MR^3}{3}, \quad A = \frac{M}{3}$$

$$\psi_m(\vec{r}) = \begin{cases} \frac{1}{3} M z & r < R \\ \frac{1}{3} MR^3 \frac{\cos \theta}{r^2} & r > R \end{cases}$$

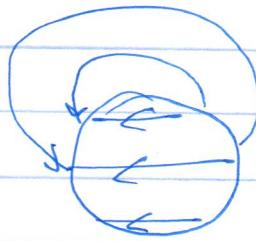
10)

$$\Rightarrow \vec{H}_m = -\nabla \Phi_m = \begin{cases} -\frac{1}{3} \vec{\mu} & r < R \\ \frac{R^3}{3} \left\{ \frac{(F \cdot \vec{\mu}) \cdot F - \vec{\mu}}{r^3} \right\} & r > R \end{cases}$$

$$\Rightarrow \vec{B}_m(F) = \begin{cases} \frac{2}{3} \mu_0 \vec{\mu} & r < R \\ \mu_0 \vec{H}_m(F) & r > R \end{cases}$$



$\vec{B}_m$  has closed field lines  $\nabla \cdot \vec{B}_m = 0$



$\vec{H}_m, \vec{B} \cdot \vec{H}_m = \sigma^* \neq 0$ !

### Ch. 13.5 The total Magnetic Field

magnetization current density  $\vec{j}_M = \nabla \times \vec{\mu}$

$\vec{j}_f$  are other sources of magnetic field: "free" currents.

Now:  $\vec{B}$  is total field produced by  $\vec{J} = \vec{j}_M + \vec{j}_f$

$$\nabla \times \vec{B} = \mu_0 \vec{J} = \mu_0 (\vec{j}_M + \vec{j}_f) = \mu_0 (\nabla \times \vec{\mu} + \vec{j}_f)$$

Define the field  $\vec{H}(F)$ :  $\vec{B}(F) = \mu_0 (\vec{H}(F) + \vec{\mu}(F))$

$$\Rightarrow \boxed{\nabla \times \vec{H} = \vec{j}_f}, \quad \nabla \cdot \vec{B} = 0 \Rightarrow \nabla \cdot \vec{H} = -\nabla \cdot \vec{\mu} = \rho^*$$

$$\text{Helmholtz} \Rightarrow \vec{H}(F) = \frac{1}{4\pi} \int d^3 r' \frac{\vec{j}_f(\vec{r}') \times (F - \vec{r}')}{|F - \vec{r}'|^3} - \nabla \Phi_M(F)$$

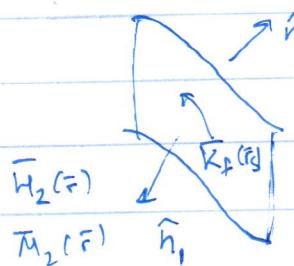
$$\Phi_M(F) = \frac{1}{4\pi} \int \frac{d^3 r' \rho^*(r')}{|F - \vec{r}'|^3} + \frac{1}{4\pi} \int dS \frac{\sigma^*(\vec{r})}{|\vec{F} - \vec{r}|}$$

Why do we get the surface term?  $\Rightarrow$  in Helm. The

$\int d^3 r \frac{\rho'(r')}{|\vec{F} - \vec{r}'|}$ ,  $\rho' = -\nabla \cdot \vec{\mu}$ , here  $\nabla \cdot \vec{\mu}$  does contain the surface term with the  $\delta$ -function on a surface.

### 13.5.1 Matching Conditions

(11)



$$\nabla \times \vec{H} = \vec{j}_F \Rightarrow \hat{n}_2 \times [\vec{H}_1 - \vec{H}_2] = \vec{K}_F$$

$$\hat{n}_2 \cdot [\vec{H}_1 - \vec{H}_2] = [\vec{\mu}_2 - \vec{\mu}_1] \cdot \hat{n}_2$$

$$\vec{n}_2 \cdot [\vec{B}_1 - \vec{B}_2] = 0$$

$\vec{A}(r)$  is continuous on  $\vec{l}_S$ .

### 13.5.2 Constitutive relations

$$M_i = \chi_{ij} H_j + \chi_{ijk}^{(a)} H_j H_k + \dots$$

### 13.6 Simple Magnetic Matter

$$\vec{M} = \chi_m \vec{H}, \quad \chi_m \text{ is magnetic susceptibility}$$

$$\vec{B} = \mu_0 [\vec{H} + \vec{M}] \Rightarrow \vec{B} = \mu \vec{H} = \chi_m \mu_0 \vec{H} = \mu_0 (1 + \chi_m) \vec{H}$$

$\mu$  is magnetic permeability

$$\chi_m = \frac{\mu}{\mu_0} \text{ relative magnetic permeability.}$$

$$\boxed{\chi_m = 1 + \chi_m}, \quad \boxed{\mu = \chi_m \mu_0}$$

For  $\chi_m > 0$   $\vec{M} \parallel \vec{H} \parallel \vec{B}$  paramagnets

$$\chi_m < 0 \quad \vec{M} \parallel \vec{H} \quad (\text{it can be shown } \mu \geq 0) \quad \vec{H} \parallel \vec{B}$$

diamagnetism is usually weak.  $(\chi_m) \sim 10^{-4} - 10^{-6}$

soft iron;  $\chi_m \sim 10^4$   
silicon steel:

#### Example 13.2

(a)

$\mu$  is permeability

Find induced

$$\vec{B}_0 \quad \vec{m}$$

$$\vec{m} = ?$$

Find the magnetic dipole moment induced by a constant magnetic field  $\vec{B}_0$  (external) in a sphere of radius  $a$  and permeability  $\mu$ .

Let's assume magnetization to be uniform

$$\vec{m} = \vec{M}V, \quad V = \frac{4}{3}\pi a^3$$

According to Application 13.1 (see page 9)

The sphere with magnetization  $\vec{M}$  produces the fields

$$\vec{H}_A(r < a) = -\frac{1}{3}\vec{M}, \quad \vec{B}_M(r < a) = \frac{2}{3}\mu_0\vec{M}$$

The total field inside the sphere

$$\vec{B}_{in} = \vec{B}_0 + \vec{B}_M(r < a) = \vec{B}_0 + \frac{2}{3}\mu_0\vec{M}$$

$$\vec{H}_{in} = \vec{H}_0 + \vec{H}_M(r < a) = \vec{H}_0 - \frac{1}{3}\vec{M}$$

We know  $\vec{B}_{in} = \mu \vec{H}_{in}$ ,  $\vec{B}_0 = \mu_0 \vec{H}_0$

$$\vec{B}_0 + \frac{2}{3}\mu_0\vec{M} = \mu \left( \frac{\vec{B}_0}{\mu_0} - \frac{1}{3}\vec{M} \right)$$

$$\vec{M} \left( \frac{2}{3}\mu_0 + \frac{1}{3}\mu \right) = \vec{B}_0 \left( \frac{\mu}{\mu_0} - 1 \right)$$

$$\vec{M} = \vec{B}_0 \frac{\frac{\mu/\mu_0 - 1}{\frac{2}{3}\mu_0 + \frac{1}{3}\mu}}{\frac{2}{3}\mu_0 + \frac{1}{3}\mu} = \frac{3}{\mu_0} \left( \frac{\mu - \mu_0}{\mu + 2\mu_0} \right) \vec{B}_0$$

$$\text{sign}(\mu - \mu_0) = \text{sign}(\mu/\mu_0 - 1) = \text{sign}(1 + \chi_m^{-1}) = \text{sign} \chi_m$$

$$\chi_m < 0 \quad \vec{M} \nparallel \vec{B}_0, \quad \chi_m > 0, \quad \vec{M} \parallel \vec{B}_0$$

## Ch. 13.6.2. Fields and Sources in Simple Magnetic Matter.

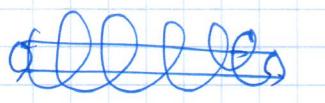
$$\vec{D} \times \vec{H} = \vec{J}_f \Rightarrow \vec{D} \times \frac{\vec{B}}{\mu} \Rightarrow \vec{D} \times \vec{B} = \mu \vec{J}_f, \vec{D} \cdot \vec{B} = 0$$

for constant  $\mu$ ,  $\Rightarrow$  some equations

as in a free space with the replacement

$$\vec{J} \rightarrow \mu \vec{J}_f$$

Example Infinite solenoid with magnetizable rod



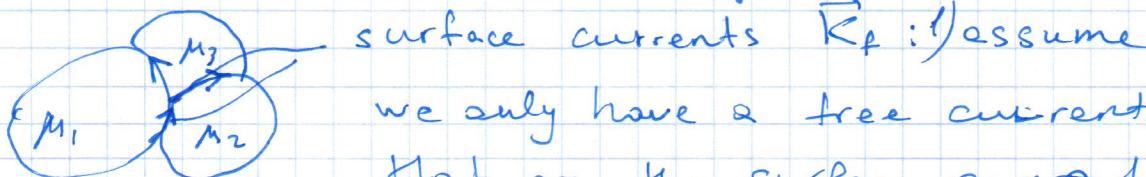
$$\vec{B} = \mu n I \hat{z}$$

as if  $n$  is # of Loops per unit length

the surface current  $K = nI$

as  $\chi_m = \mu/\mu_0$  can be very large ( $10^5$ ) this change is substantial.

## Ch. opter 13.6.4 Potential Theory of a simple Magnet



surface currents  $\vec{K}_f$ : assume

we only have a free currents  
that are the surface currents

flowing over the interfaces

2) assume  $\mu_i$  is constant in each of the regions.

$$\vec{H} = -\vec{\nabla} \psi \quad (\vec{\nabla} \times \vec{H} = 0 \text{ away from the interfaces!})$$

in each of the regions not on a surface!

$$\text{away from the interface also } \vec{\nabla} \cdot \vec{B} = \vec{\nabla} \cdot \frac{\vec{H}}{\mu_i} \stackrel{i}{=} \frac{1}{\mu_i} \vec{\nabla} \cdot \vec{H} = 0$$

$\vec{H} = -\vec{\nabla} \psi, \vec{\nabla}^2 \psi = 0 \Rightarrow$  simple electrostatic equation  
inside each region.

(14)

Let's formulate the boundary conditions.

$$\vec{\nabla} \times \vec{H} = \vec{K} \Rightarrow \hat{n}_2 \times [\vec{H}_1 - \vec{H}_2] = \vec{K}_L$$

If  $\vec{K}_L = 0$ !  $\Rightarrow \vec{\nabla}^2 \psi = 0 \Rightarrow \psi_1(\vec{r}_S) = \psi_2(\vec{r}_S)$

$$\Downarrow \vec{n} = \vec{H}_m = -\vec{\nabla} \psi_m, \quad \psi_m = \frac{1}{4\pi} \int d^3 r' \frac{\rho^*(r')}{|\vec{r} - \vec{r}'|} + \frac{1}{4\pi} \int dS \frac{\epsilon^*(r_S)}{|\vec{r} - \vec{r}_S|}$$

$$\vec{H} = \vec{H}_m = -\vec{\nabla} \psi_m \Rightarrow \psi_{m_1}(\vec{r}_S) = \psi_{m_2}(\vec{r}_S) \Rightarrow \psi_1(\vec{r}_S) = \psi_2(\vec{r}_S)$$

$\uparrow K_L \neq 0$

$$\vec{\nabla} \cdot \vec{B} = 0 \Rightarrow \mu_1 \left. \frac{\partial \psi_1}{\partial n} \right|_S = \mu_2 \left. \frac{\partial \psi_2}{\partial n} \right|_S$$

A Magnetizable rod in a transverse External Field



$$\psi_0 = -\left(B_0/\mu_0\right)x = -H_0 p \cos \theta$$

$$\psi(r, \theta) = \begin{cases} Ap \cos \theta & r < R \\ (C/p + H_0 p) \cos \theta & r > R \end{cases}$$

$$\psi_1 = \psi_2 \Rightarrow AR = C/R - H_0 R, \mu \frac{\partial \psi_1}{\partial p} = \mu_0 \frac{\partial \psi_2}{\partial p}$$

$$\mu A = \mu_0 \left( -\frac{C}{R^2} - H_0 \right) \Rightarrow \text{odd}$$

$$\mu_0 A = \mu_0 \left( \frac{C}{R^2} - H_0 \right) \quad (\mu + \mu_0) A = -2\mu_0 H_0$$

$$A = -\frac{2\mu_0}{\mu + \mu_0} H_0$$

$$\mu_0 \mu A = \mu_0^2 \left( -\frac{c}{R^2} - H_0 \right) \Rightarrow \text{subtract}$$

(15)

$$\mu \mu_0 A = \mu_0 \mu \left( \frac{c}{R^2} - H_0 \right)$$

$$\frac{c}{R^2} (\mu_0 \mu + \mu_0^2) = H_0 (\mu_0 \mu - \mu_0^2)$$

$$C = \frac{\mu - \mu_0}{\mu + \mu_0} R^2 H_0$$

outside the rod,  $\vec{B}(r) = -\mu_0 \nabla \frac{1}{4\pi r}$  is the sum of a uniform and dipole field

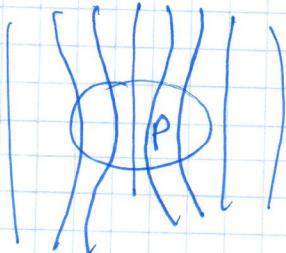
inside the <sup>rod</sup> ~~induced~~ field is

$$\vec{B}_{in} = -\hat{x} A \mu = -\hat{x} \mu \frac{-2\mu_0}{\mu + \mu_0} H_0 = \vec{B}_0 \frac{2\mu}{\mu + \mu_0}$$

$$\boxed{\vec{B}_{in} = \vec{B}_0 \frac{2\chi_m}{\chi_m + 1}}$$

$\vec{B}_{in} > \vec{B}_0$   $\chi_m > 1$  ( $\chi > 0$  paramagnet)  $\chi = 1 + \chi'$

$\vec{B}_{in} < \vec{B}_0$   $\chi_m < 1$  ( $\chi < 0$  diamagnet)



superconductor  $\chi = -1$

