Examples of Dirichlet Problems

The boundary value problem we are dealing with is to find the potential $\Phi(\boldsymbol{x})$ satisfying

$$\nabla^2\Phi(\boldsymbol{x}) = -\frac{\rho(\boldsymbol{x})}{\epsilon_0}\,, \boldsymbol{x} \in V, \quad \Phi(\boldsymbol{x}) = \Phi_0(\boldsymbol{x})\,, \boldsymbol{x} \in S \tag{1}$$

where V is a given volume (finite or infinite), and S is the surface bounding the volume V. The surface may or may not include infinity and may or may not have infinite pieces. The charge density, $\rho(\boldsymbol{x})$ and the potential on the surface, Φ_S are prescribed functions on in a volume and on a surface.

One method to solve (1) is to find the Green function first. The Green function, $G(\boldsymbol{x}|\boldsymbol{x}')$ is itself a solution of a particular Dirichlet problem,

$$\nabla^2\Phi(\boldsymbol{x}) = -4\pi\delta(\boldsymbol{x}-\boldsymbol{x}')\,, \boldsymbol{x}, \boldsymbol{x}' \in V, \quad \Phi(\boldsymbol{x}) = 0\,, \boldsymbol{x} \in S \tag{2}$$

which physically corresponds to placing the point charge of a magnitude $Q = 4\pi\epsilon_0$ at a location $\boldsymbol{x}'$ in a volume V and grounding the surface electrodes.

Once the Green function defined by (2) is found the solution to *any* Dirichlet problem is obtained as follows,

$$\Phi(\boldsymbol{x}) = \frac{1}{4\pi\epsilon_0}\int d^3\boldsymbol{x}'\rho(\boldsymbol{x}')G(\boldsymbol{x}|\boldsymbol{x}') + \frac{1}{4\pi}\oint_S da'\Phi(\boldsymbol{x}')\partial_{n'}G(\boldsymbol{x}|\boldsymbol{x}') \tag{3}$$

In this note I show some examples of finding and using the Green functions to solve the Dirichlet problems. In some instances the comparison is made with other approaches.

I. GREEN FUNCTION FOR THE WHOLE SPACE

In this case the volume V is the whole space. And the surface, S is at infinity. The solution of (2) that vanishes at infinity is provided by a Coulomb law,

$$G(\boldsymbol{x}|\boldsymbol{x}') = \frac{1}{4\pi\epsilon_0}\frac{Q}{|\boldsymbol{x}-\boldsymbol{x}'|} = \frac{1}{|\boldsymbol{x}-\boldsymbol{x}'|} \tag{4}$$

and as a result the solution of (1) that is finite and regular at infinity reads,

$$\Phi(\boldsymbol{x}) = \frac{1}{4\pi\epsilon_0}\int d^3\boldsymbol{x}'\frac{\rho(\boldsymbol{x}')}{|\boldsymbol{x}-\boldsymbol{x}'|} \tag{5}$$

as expected. Observe that the Green function (4) is symmetric in $\boldsymbol{x}$ and $\boldsymbol{x}'$ interchange as is generally true. We now turn to slightly more complicated cases.

II. DIRICHLET PROBLEM IN THE REGION $x > 0$

In this section we study the Dirichlet problem in the half-space which is next simplest case after the whole space considered above. I find the Green function in the region $x > 0$ employing different approaches. This is meant to demonstrate how these different strategies are applied.

The section is structured as follows. I start with finding the Green function for the region $x > 0$ by the method of images in Sec. II A. Next, I find the Green function by the method of direct integration in Sec. II B. Then in Sec. II C I write down the general solution using the Green function computed previously. I next check that what we got by the method of Green function is indeed a solution in Sec. II C 1. And finally I solve the problem by following the alternative method of separation of variables in Sec. II D.

A. Green function by the method of image charges

As the half-space is rather simple we can try to solve the boundary value problem (2) by the method of images. What we have here is an elementary problem of finding the potential in a half-space bounded by an infinite grounded plane once a point charge is placed somewhere in the half-space. We now how to solve this problem by the method of images,

$$G(\boldsymbol{x}|\boldsymbol{x}') = \frac{1}{4\pi\epsilon_0}\left[\frac{Q}{|\boldsymbol{x}-\boldsymbol{x}'|} - \frac{Q}{|\boldsymbol{x}-\boldsymbol{x}'_I|}\right], \tag{6}$$

where the image charge is located at

$$\boldsymbol{x}'_I = (-x', y', z') \tag{7}$$

Then by symmetry $G(\boldsymbol{x}|\boldsymbol{x}') = 0$ for $x' = 0$ and as image is outside of the region of interest, $x > 0$ we have (2) satisfied. The relation (6) in explicite form reads

$$G(x,y,z|x',y',z') = \frac{1}{\sqrt{(y'-y)^2+(z'-z)^2+(x-x')^2}} - \frac{1}{\sqrt{(y'-y)^2+(z'-z)^2+(x+x')^2}} \tag{8}$$

B. Green function by the method of direct integration

We now illustrate how to use the method of direct integration to obtain Green function. After all the Green function is just a solution to one very specific boundary value problem. This method can therefore be viewed as a modification of the separation of variables approach to the particular boundary condition as well as to the singular source as inhomogeneity. So, whenever the separation of variables works (i.e. when the geometry is simple enough) it could be used to find analytical expressions for the Green function in the form of an integral or infinite series or sometimes both. And that is what we do now for the half-space problem. The Green function satisfies

$$\nabla^2_{\boldsymbol{x}'} G(x,y,z|x',y',z') = -4\pi\delta(x-x')\delta(y-y')\delta(z-z'), \quad G(x,y,z|x'=0,y',z') = 0 \tag{9}$$

$$G(x,y,z|x',y',z') = \int \frac{dk_y}{2\pi}\int\frac{dk_z}{2\pi} e^{ik_y y' + ik_z z'}\bar{g}(x,y,z|x',k_y,k_z) \tag{10}$$

$$\int \frac{dk_y}{2\pi}\int\frac{dk_z}{2\pi} e^{ik_y y' + ik_z z'}(-k_y^2 - k_z^2 + \partial^2_{x'})\bar{g}(x,y,z|x',k_y,k_z) = -4\pi\delta(x-x')\int\frac{dp_y}{2\pi}\int\frac{dp_z}{2\pi}e^{ip_y(y'-y)+ip_z(z'-z)} \tag{11}$$

which holds for all y' and z', therefore,

$$(-k_y^2 - k_z^2 + \partial^2_{x'})\bar{g}(x,y,z|x',k_y,k_z) = -4\pi\delta(x-x')e^{ik_y(-y)+ik_z(-z)} \tag{12}$$

Define

$$\bar{g}(x,y,z|x',k_y,k_z) = g(x,y,z|x',k_y,k_z)e^{ik_y(-y)+ik_z(-z)} \tag{13}$$

such that

$$G(x,y,z|x',y',z') = \int \frac{dk_y}{2\pi}\int\frac{dk_z}{2\pi} e^{ik_y(y'-y)+ik_z(z'-z)}g(x,y,z|x',k_y,k_z) \tag{14}$$

$$(-k_y^2 - k_z^2 + \partial^2_{x'})g(x,y,z|x',k_y,k_z) = -4\pi\delta(x-x') \tag{15}$$

It follows that $\bar{g}$ is independent of y and z. We can take the corresponding derivatives of (15) to show that. Hence $g(x,y,z|x',k_y,k_z) \equiv g(x|x',k_y,k_z)$, and

$$(-k_y^2 - k_z^2 + \partial^2_{x'})g(x|x',k_y,k_z) = -4\pi\delta(x-x') \tag{16}$$

g has a cusp discontinuity as a function of x' at $x = x'$, which by integration,

$$\partial_{x'}g(x|x',k_y,k_z)|_{x'=x+\epsilon} - \partial_{x'}g(x|x',k_y,k_z)|_{x'=x-\epsilon} = -4\pi \tag{17}$$

The contineous solution satisfying the boundary condition reads with $k = \sqrt{k_y^2 + k_z^2}$

$$g(x|x', k_y, k_z) = A(k)\sinh(kx_<)e^{-kx_>} = \begin{cases} A(k)\sinh(kx')e^{-kx} & 0 < x' < x \\ A(k)\sinh(kx)e^{-kx'} & x' > x \end{cases} \tag{18}$$

Eq. (17) gives

$$A(k)[-ke^{-kx}\sinh(kx) - k\cosh(kx)e^{-kx}] = -4\pi \tag{19}$$

$$A(k) = \frac{-4\pi e^{kx}}{-k\sinh(kx) - k\cosh(kx)} = \frac{4\pi}{k} \tag{20}$$

Eq (14) gives

$$G(x,y,z|x',y',z') = \int \frac{dk_y}{2\pi} \int \frac{dk_z}{2\pi} e^{ik_y(y'-y)+ik_z(z'-z)} \frac{4\pi}{k} \sinh(kx_<)e^{-kx_>} \tag{21}$$

Polar coordinates,

$$G(x,y,z|x',y',z') = (2\pi)^{-2} \int_0^{2\pi} d\phi \int_0^{\infty} kdke^{ik(y'-y)\cos\phi + ik(z'-z)\sin\phi} \left(\frac{4\pi}{k}\right) \frac{1}{2}(e^{-kx_>+kx_<} - e^{-kx_>-kx_<})$$
$$G(x,y,z|x',y',z') = 4\pi(2\pi)^{-1} \int_0^{\infty} dkJ_0[k\sqrt{(y'-y)^2+(z'-z)^2}]\frac{1}{2}(e^{-kx_>+kx_<} - e^{-kx_>-kx_<}) \tag{22}$$

Use the integral

$$\int_0^{\infty} d\zeta J_0(\zeta a)e^{-\zeta b} = \frac{1}{\sqrt{a^2+b^2}} \tag{23}$$

$$\begin{aligned} G(x,y,z|x',y',z') &= \frac{1}{\sqrt{(y'-y)^2+(z'-z)^2+(x_>-x_<)^2}} - \frac{1}{\sqrt{(y'-y)^2+(z'-z)^2+(x_>+x_<)^2}} \\ &= \frac{1}{\sqrt{(y'-y)^2+(z'-z)^2+(x-x')^2}} - \frac{1}{\sqrt{(y'-y)^2+(z'-z)^2+(x+x')^2}} \end{aligned} \tag{24}$$

In agreement with the charge and its image.

C. General solution of the Dirichlet boundary value problem in half-space with Green function

Having obtained the Green function we can write the general solution of the Dirichlet boundary value problem in half-space. From (3) we obtain (omitting the volume density first term and keeping only the second for clarity),

$$\nabla^2\Phi(x,y,z) = 0, x > 0; \quad \Phi(x=0,y,z) = \Phi_0(y,z) \tag{25}$$

then reads

$$\Phi(x,y,z) = \frac{1}{4\pi} \int dy'dz'\Phi_0(y',z') \left[\frac{\partial}{\partial x'} G(x,y,z|x',y',z')\right]_{x'=0} \tag{26}$$

Here the $''-''$ in front got cancelled as $\partial/\partial n' = -\partial/\partial x'$.

$$\Phi(x,y,z) = \frac{1}{2\pi} \int dy'dz'\Phi_0(y',z') \frac{x}{(x^2+(y-y')^2+(z-z')^2)^{3/2}} \tag{27}$$

1. *Direct check of the solution* (27)

Let us directly check that (27) solves the Dirichlet problem in the half space. As we have no charges in the region of interest in this case (27) has to satisfy the Laplace equation anywhere in the region $x > 0$. Let's check this,

$$\begin{aligned}
\nabla^2_{\boldsymbol{x}}\Phi(x,y,z) &= \nabla^2_{\boldsymbol{x}}[\frac{1}{2\pi}\int dy'dz'\Phi_0(y',z')\frac{x}{(x^2+(y-y')^2+(z-z')^2)^{3/2}}] \\
&= -\nabla^2_{\boldsymbol{x}}\frac{\partial}{\partial x}[\frac{1}{2\pi}\int dy'dz'\Phi_0(y',z')\frac{1}{(x^2+(y-y')^2+(z-z')^2)^{1/2}}] \\
&= -\frac{\partial}{\partial x}\frac{1}{2\pi}\int dy'dz'\Phi_0(y',z')\nabla^2_{\boldsymbol{x}}[\frac{1}{(x^2+(y-y')^2+(z-z')^2)^{1/2}}] \\
&= (4\pi)\frac{\partial}{\partial x}\frac{1}{2\pi}\int dy'dz'\Phi_0(y',z')\delta(x)\delta(y-y')\delta(z-z') = 0
\end{aligned} \tag{28}$$

because $x \neq 0$ everywhere in the region $x > 0$, and therefore $\delta(x) = 0$.

Now the boundary conditions are also satisfied by (27) thanks to the representation of the two-dimensional δ-function,

$$\delta(x)\delta(y) = \frac{1}{2\pi}\lim_{\epsilon\to 0}\frac{\epsilon}{(x^2+y^2+\epsilon^2)^{3/2}} \tag{29}$$

Indeed the function (29) is peaked at $x = y = 0$, and integrates to one,

$$\frac{1}{2\pi}\int dxdy\frac{\epsilon}{(x^2+y^2+\epsilon^2)^{3/2}} = \frac{1}{2\pi}\int_0^{2\pi}d\phi\int_0^{\infty}rdr\frac{\epsilon}{(r^2+\epsilon^2)^{3/2}} = \int_0^{\infty}dx\frac{x}{(x^2+1)^{3/2}} = 1 \tag{30}$$

Thanks to the property (29)

$$\lim_{x\to 0^+}\Phi(x,y,z) = \int dy'dz'\Phi_0(y',z')\delta(y-y')\delta(z-z') = \Phi_0(y,z) \tag{31}$$

Eqs. (28) and (31) show that (27) is the solution to the Dirichlet problem.

D. Solution by separation of variables without finding Green function

Certainly, the problem in the half space is simple enough to be solvable by a standard separation of variables in Cartesian coordinates. Here we bring the solution for illustration and comparison purposes.

$$\Phi(x,y,z) = \int\frac{dk_y}{2\pi}\int\frac{dk_z}{2\pi}\Phi_{k_y,k_z}e^{ik_yy+ik_zz}e^{-kx} \tag{32}$$

where $k = \sqrt{k_y^2+k_z^2}$ and

$$\Phi_{k_y,k_z} = \int dy'dz'e^{-ik_yy'-ik_zz'}\Phi_0(y',z') \tag{33}$$

$$\Phi(x,y,z) = \int dy'dz'\Phi_0(y',z')\int\frac{dk_y}{2\pi}\int\frac{dk_z}{2\pi}e^{ik_y(y-y')+ik_z(z-z')}e^{-kx} \tag{34}$$

$$\begin{aligned}
\int\frac{dk_y}{2\pi}\int\frac{dk_z}{2\pi}e^{ik_y(y-y')+ik_z(z-z')}e^{-kx} &= (2\pi)^{-2}\int_0^{2\pi}d\phi\int_0^{\infty}kdke^{ik(y'-y)\cos\phi+ik(z'-z)\sin\phi}e^{-kx} \\
&= (2\pi)^{-2}(2\pi)\int_0^{\infty}kdkJ_0(k\sqrt{(y-y')^2+(z-z')^2})e^{-kx}
\end{aligned} \tag{35}$$

Employ the relation

$$\int_0^\infty d\eta\eta J_0(\eta s)e^{-\eta w} = \frac{w}{(s^2+w^2)^{3/2}} \tag{36}$$

to obtain

$$\begin{aligned}\int \frac{dk_y}{2\pi}\int \frac{dk_z}{2\pi}e^{ik_y(y-y')+ik_z(z-z')}e^{-kx} &= \frac{1}{2\pi}\int_0^\infty kdkJ_0(k\sqrt{(y-y')^2+(z-z')^2})e^{-kx}\\ &\frac{1}{2\pi}\frac{x}{(y-y')^2+(z-z')^2+x^2)^{3/2}}\end{aligned} \tag{37}$$

Substituting (37) in (34) reproduces the solution (27) which was obtained by the method of Green functions.

III. DIRICHLET PROBLEM IN THE REGION IN BETWEEN THE TWO CONCENTRIC SPHERES: SOLUTION BY THE METHOD OF DIRECT INTEGRATION

Consider the region of interest being set by $a < r < b$. We have to find the solution of the following problem,

$$\nabla_x^2 G(\boldsymbol{x},\boldsymbol{x}') = -4\pi\delta(\boldsymbol{x}-\boldsymbol{x}')\,, \quad \& \; G(\boldsymbol{x},\boldsymbol{x}') = 0, |\boldsymbol{x}| = a, b \tag{38}$$

Look for the Green function in the form,

$$G(\boldsymbol{x},\boldsymbol{x}') = \sum_{l=0}^{\infty}\sum_{m=-l}^{l} A_{lm}(r|\underbrace{r'\theta'\phi'}_{\boldsymbol{x}'})Y_{lm}(\theta\phi) \tag{39}$$

where $\boldsymbol{x}'$ is a parameter.

Write

$$\delta(\boldsymbol{x}-\boldsymbol{x}') = \frac{1}{r^2}\delta(r-r')\delta(\phi-\phi')\delta(\cos\theta-\cos\theta') \tag{40}$$

Use completeness, (C7),

$$\delta(\boldsymbol{x}-\boldsymbol{x}') = \frac{1}{r^2}\delta(r-r')\sum_{l=0}^{\infty}\sum_{m=-l}^{l} Y^*_{lm}(\theta'\phi')Y_{lm}(\theta\phi) \tag{41}$$

Apply Laplace operator to (39) and use the property (C3)

$$\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\frac{1}{r}\frac{\partial^2}{\partial r^2}[rA_{lm}(r|\underbrace{r'\theta'\phi'}_{\boldsymbol{x}'})Y_{lm}(\theta\phi)] - \frac{l(l+1)}{r^2}A_{lm}(r|\underbrace{r'\theta'\phi'}_{\boldsymbol{x}'})Y_{lm}(\theta\phi) = -4\pi\frac{1}{r^2}\delta(r-r')\sum_{l=0}^{\infty}\sum_{m=-l}^{l} Y^*_{lm}(\theta'\phi')Y_{lm}(\theta\phi) \tag{42}$$

The expansion of G in $Y_{lm}(\theta\phi)$ is *unique*, therefore for *all* values of l and m in (42),

$$\frac{1}{r}\frac{\partial^2}{\partial r^2}[rA_{lm}(r|\underbrace{r'\theta'\phi'}_{\boldsymbol{x}'})] - \frac{l(l+1)}{r^2}A_{lm}(r|\underbrace{r'\theta'\phi'}_{\boldsymbol{x}'}) = -4\pi\frac{1}{r^2}\delta(r-r')Y^*_{lm}(\theta'\phi') \tag{43}$$

The functions $Y^*_{lm}(\theta'\phi')$ form the complete set as well as $Y_{lm}(\theta'\phi')$ because of the property (C2). Therefore the expansion of $A_{lm}(r|\underbrace{r'\theta'\phi'}_{\boldsymbol{x}'})$ for any given l, m contains only *one* term,

$$A_{lm}(r|\underbrace{r'\theta'\phi'}_{\boldsymbol{x}'}) = g_{lm}(r,r')Y^*_{lm}(\theta'\phi')\,, \tag{44}$$

where the expansion coefficient satisfies the differential equation,

$$\frac{1}{r}\frac{d^2}{dr^2}[rg_{lm}(r,r')] - \frac{l(l+1)}{r^2}g_{lm}(r,r') = -\frac{4\pi}{r^2}\delta(r-r') \tag{45}$$

It follows from (45) that $g_{lm}(r,r')$ is the same for *all* values of $m = -l, \ldots, +l$, and we write $g_{lm}(r,r') \equiv g_l(r,r')$. The general solution of (45) reads,

$$g_l(r,r') = \begin{cases} Ar^l + Br^{-(l+1)}, & r < r' \\ A'r^l + B'r^{-(l+1)}, & r > r' \end{cases} \tag{46}$$

In the Dirichlet problem we must impose the condition,

$$g_l(a,r') = g_l(b,r') = 0 \tag{47}$$

So (46) becomes,

$$g_l(r,r') = \begin{cases} A\left(r^l - \frac{a^{2l+1}}{r^{l+1}}\right), & r < r' \\ B'\left(\frac{1}{r^{l+1}} - \frac{r^l}{b^{2l+1}}\right), & r > r' \end{cases} \tag{48}$$

The functions $g_l(r,r')$ with r' being a parameter and r an argument that are continuous at $r = r'$ can be written as

$$g_l(r,r') = C\left(r_<^l - \frac{a^{2l+1}}{r_<^{l+1}}\right)\left(\frac{1}{r_>^{l+1}} - \frac{r_>^l}{b^{2l+1}}\right) \tag{49}$$

where as usual, $r_> \equiv \max\{r,r'\}$, and $r_< \equiv \min\{r,r'\}$.

It remains to fix the constant C in (49). To this end apply the operation, $\lim_{\epsilon\to 0}\int_{r'-\epsilon}^{r'+\epsilon} drr$ to (45). which gives the matching condition,

$$\left\{\frac{d}{dr}\left[rg_l(r,r')\right]\right\}_{r'+\epsilon} - \left\{\frac{d}{dr}\left[rg_l(r,r')\right]\right\}_{r'-\epsilon} = -\frac{4\pi}{r'} \tag{50}$$

Using (49) we compute the derivatives appearing in (50). For $r = r' + \epsilon$, $r > r'$,

$$\begin{aligned} \left\{\frac{d}{dr}\left[rg_l(r,r')\right]\right\}_{r'+\epsilon} &= C\left((r')^l - \frac{a^{2l+1}}{(r')^{l+1}}\right)\left[\frac{d}{dr}\left(\frac{1}{r^l} - \frac{r^{l+1}}{b^{2l+1}}\right)\right]_{r\to r'} \\ &= -\frac{C}{r'}\left[1 - \left(\frac{a}{r'}\right)^{2l+1}\right]\left[l + (l+1)\left(\frac{r'}{b}\right)^{2l+1}\right] \\ &= -\frac{C}{r'}\left(l - (l+1)\left(\frac{a}{b}\right)^{2l+1}\right) + \frac{C}{r'}l\left(\frac{a}{r'}\right)^{2l+1} - \frac{C}{r'}(l+1)\left(\frac{r'}{b}\right)^{2l+1} \end{aligned} \tag{51}$$

In the same way,

$$\begin{aligned} \left\{\frac{d}{dr}\left[rg_l(r,r')\right]\right\}_{r'-\epsilon} &= C\left[\frac{d}{dr}\left(r^{l+1} - \frac{a^{2l+1}}{r^l}\right)\right]_{r\to r'}\left(\frac{1}{(r')^{l+1}} - \frac{(r')^l}{b^{2l+1}}\right) \\ &= C\left[(l+1)(r')^l + l\frac{a^{2l+1}}{(r')^{l+1}}\right]\left[\frac{1}{(r')^{l+1}} - \frac{(r')^l}{b^{2l+1}}\right] \\ &= \frac{C}{r'}\left[(l+1) + l\left(\frac{a}{r'}\right)^{2l+1}\right]\left[1 - \left(\frac{r'}{b}\right)^{2l+1}\right] \\ &= \frac{C}{r'}\left((l+1) - l\left(\frac{a}{b}\right)^{2l+1}\right) + \frac{C}{r'}l\left(\frac{a}{r'}\right)^{2l+1} - \frac{C}{r'}(l+1)\left(\frac{r'}{b}\right)^{2l+1} \end{aligned} \tag{52}$$

Subtract (52) from (51) and substitute to (50),

$$-\frac{C}{r'}\left[(2l+1) - (2l+1)\left(\frac{a}{b}\right)^{2l+1}\right] = -\frac{4\pi}{r'} \tag{53}$$

Then,

$$C = \frac{4\pi}{(2l+1)\left[1 - \left(\frac{a}{b}\right)^{2l+1}\right]} \tag{54}$$

Substitute (54) in (49) to get,

$$g_l(r,r') = \frac{4\pi}{(2l+1)\left[1-\left(\frac{a}{b}\right)^{2l+1}\right]}\left(r_<^l - \frac{a^{2l+1}}{r_<^{l+1}}\right)\left(\frac{1}{r_>^{l+1}} - \frac{r_>^l}{b^{2l+1}}\right) \tag{55}$$

Substituting (55) in (44) and finally to (39)

$$G_D(\boldsymbol{x},\boldsymbol{x}') = 4\pi\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\frac{Y_{lm}^*(\theta'\phi')Y_{lm}(\theta\phi)}{(2l+1)\left[1-\left(\frac{a}{b}\right)^{2l+1}\right]}\left(r_<^l - \frac{a^{2l+1}}{r_<^{l+1}}\right)\left(\frac{1}{r_>^{l+1}} - \frac{r_>^l}{b^{2l+1}}\right) \tag{56}$$

Appendix A: Some properties of the Legendre polynomials

There are different ways to introduce these polynomials and we follow [**?**]. Consider the two vectors, $\boldsymbol{r} = r\hat{r}$ and $\boldsymbol{r}' = \hat{r'}r'$. For $r' < r$ define $r'/r = \rho$, $|\rho| < 1$. Then,

$$\frac{1}{|\boldsymbol{r}-\boldsymbol{r}'|} = \frac{1}{r}\sum_{l=0}^{\infty}\rho^l P_l(\cos\gamma) \tag{A1}$$

where $\cos\gamma = \hat{r}\cdot\hat{r'}$. Lets identify few first polynomials, using the Taylor expansion,

$$\frac{1}{\sqrt{1+x}} = 1 - \frac{x}{2} + \frac{3x^2}{8} - \frac{5x^3}{16} + \dots \tag{A2}$$

$$\frac{1}{|\boldsymbol{r}-\boldsymbol{r}'|} = \frac{1}{r}\frac{1}{\sqrt{1-2\rho\cos\gamma+\rho^2}} = \frac{1}{r}\left[1-(1/2)(-2\rho\cos\gamma+\rho^2)+\frac{3}{8}(-2\rho\cos\gamma+\rho^2)^2-\frac{5}{16}(-2\rho\cos\gamma+\rho^2)^3+\dots\right] \tag{A3}$$

So that we read off,

$$P_0(x) = 1\,,\quad P_1(x) = x,\quad P_2(x) = -\frac{1}{2}+\frac{3}{2}x^2,\quad P_3(x) = -\frac{3}{2}x+\frac{5}{2}x^3 \tag{A4}$$

By construction the functions, $r'^l P_l(\cos\gamma)$ must be the polynomial solutions to the Laplace equation, $\nabla^2_{\boldsymbol{r}'}[r'^l P_l(\cos\gamma)] = 0$ that have an additional property of being symmetric relative to the $\frac{\boldsymbol{r}-\boldsymbol{r}'}{|\boldsymbol{r}-\boldsymbol{r}'|}$ spatial direction. There are infinitely many solutions symmetric relative to the specific axis. They can be shown to all be of the form of the series in $r'^l P_l(\cos\gamma)$ but with coefficients that can be arbitrary, and not just equal to unity as in (A1). The polynomials defined in this way, satisfy the condition,

$$P_l(x=1) = 1 \tag{A5}$$

because for $\cos\gamma = 1$,

$$\frac{1}{|\boldsymbol{r}-\boldsymbol{r}'|} = \frac{1}{r}\frac{1}{\sqrt{1-2\rho+\rho^2}} = \frac{1}{r}\sum_{l=0}^{\infty}\rho^l = \frac{1}{r}\sum_{l=0}^{\infty}\rho^l P_l(1) \tag{A6}$$

It is also clear that the generic term in the expansion (A1) is of the form $(\rho\cos\gamma)^j\rho^{2k}$ and the polynomials P_l for even (odd) l are even (odd) functions of their argument,

$$P_l(-x) = (-1)^l P_l(x) \tag{A7}$$

Alternative definition is via the solutions of the differential equation,

$$\frac{d}{dx}\left[(1-x^2)\frac{dP}{dx}\right] + l(l+1)P = 0 \tag{A8}$$

that are regular in the interval $|x| \leq 1$ (including $x = \pm 1!$) Such solutions to the Legendre equation, (A8) exists only for $l = 0, 1, 2, \ldots$, and are the Legendre polynomials, $P_l(x)$.

Rodrigues formula,

$$P_l(x) = \frac{1}{2^l l!} \frac{d^l}{dx^l} (x^2 - 1)^l \tag{A9}$$

Orthogonality and normalization,

$$\int_{-1}^{1} dx P_l(x) P_{l'}(x) = \frac{2}{2l+1} \delta_{l,l'} \tag{A10}$$

Appendix B: Associated Legendre Functions

These functions satisfies the differential equation,

$$\frac{d}{dx}\left[(1-x^2)\frac{dP}{dx}\right] + \left[l(l+1) - \frac{m^2}{1-x^2}\right] P = 0 \tag{B1}$$

For $m \geq 0$

$$P_l^m(x) = (-1)^m (1-x^2)^{m/2} \frac{d^m}{dx^m} P_l(x) \tag{B2}$$

Using the Rodrigues formula,

$$P_l^m(x) = \frac{(-1)^m}{2^l l!} (1-x^2)^{m/2} \frac{d^{l+m}}{dx^{l+m}} (x^2-1)^l \tag{B3}$$

The polynomials $P_l^m(x)$ and $P_l^{-m}(x)$ are proportional as the defining equation depends on m^2 rather than m itself. Specifically,

$$P_l^{-m}(x) = (-1)^m \frac{(l-m)!}{(l+m)!} P_l^m(x) \tag{B4}$$

Appendix C: Some properties of spherical harmonics

- **Definition**. For a given $l = 0, 1, \ldots$, the allowed m are integers, $m = -l, -l+1, , \ldots, 0, 1, \ldots + l$ that are $2l+1$ in number. For $0 \leq m \leq l$, $Y_{lm}(\theta\phi) \propto P_l^m(\cos\theta) e^{im\phi}$,

$$Y_{lm}(\theta\phi) = \sqrt{\frac{2l+1}{4\pi} \frac{(l-m)!}{(l+m)!}} P_l^m(\cos\theta) e^{im\phi} \tag{C1}$$

The Associated Legendre polynomials are defined for $m \geq 0$, and for $m \leq 0$ we have

$$Y_{l,-m}(\theta\phi) = (-1)^m Y_{lm}^*(\theta\phi) \tag{C2}$$

- **Differential Equation**

$$\frac{1}{\sin\theta} \frac{\partial}{\partial\theta} \left(\sin\theta \frac{\partial Y_{lm}}{\partial\theta}\right) + \left[l(l+1) Y_{lm} + \frac{1}{\sin^2\theta} \frac{\partial^2 Y_{lm}}{\partial^2\phi}\right] = 0\,. \tag{C3}$$

Indeed,

$$\frac{\partial^2 Y_{lm}}{\partial^2\phi} = -m^2 Y_{lm} \tag{C4}$$

so the above property is equivalent to,

$$\frac{1}{\sin\theta} \frac{\partial}{\partial\theta} \left(\sin\theta \frac{\partial Y_{lm}}{\partial\theta}\right) + \left[l(l+1) - \frac{m^2}{\sin^2\theta}\right] Y_{lm} = 0 \tag{C5}$$

Which is trivially satisfied again since $Y_{lm}(\theta\phi) \propto P_l^m e^{im\phi}$.

- **Normalization and orthonormality**

$$\int_0^{2\pi} d\phi \int_0^{\pi} \sin\theta d\theta Y^*_{l'm'}(\theta\phi) Y_{lm}(\theta\phi) = \delta_{ll'}\delta_{mm'} \tag{C6}$$

- **Completeness**

$$\sum_{l=0}^{\infty} \sum_{m=-l}^{l} Y^*_{lm}(\theta\phi) Y_{lm}(\theta'\phi') = \delta(\phi - \phi')\delta(\cos\theta - \cos\theta') \tag{C7}$$

- **Addition Theorem**

$$P_l(\cos\gamma) = \frac{4\pi}{2l+1} \sum_{m=-l}^{m=l} Y^*_{lm}(\theta'\phi') Y_{lm}(\theta\phi)\,, \tag{C8}$$

where

$$\cos\gamma = \boldsymbol{n} \cdot \boldsymbol{n}' = \cos\theta\cos\theta' + \sin\theta\sin\theta'\cos(\phi - \phi') \tag{C9}$$

is the dot product of the two unit vectors, $\boldsymbol{n} = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$ and $\boldsymbol{n}' = (\sin\theta'\cos\phi', \sin\theta'\sin\phi', \cos\theta')$.

- **Azimuthally symmetric spherical harmonics,** $m = 0$

$$Y_{l0}(\theta, \phi) = \sqrt{\frac{2l+1}{4\pi}} P_l(\cos\theta) \tag{C10}$$