

Maxwell's Equations (Chapter 20 A.2.)

(A.2.), 20.2

EM fields produced by charges and currents

$$\nabla \cdot \vec{E} = \frac{\rho}{\epsilon_0}$$

$$\nabla \cdot \vec{B} = 0$$

$$\nabla \times \vec{E} = - \frac{\partial \vec{B}}{\partial t}$$

$$\nabla \times \vec{B} = \mu_0 \vec{J} + \frac{1}{c^2} \frac{\partial \vec{E}}{\partial t}$$

By considering the $\nabla \times (\nabla \times)$ double curl equations we can get the inhomogeneous wave equations

$$\nabla^2 \vec{E} - \frac{1}{c^2} \frac{\partial^2 \vec{E}}{\partial t^2} = \frac{1}{\epsilon_0} \nabla \rho + \mu_0 \frac{\partial \vec{J}}{\partial t}$$

$$\nabla^2 \vec{B} - \frac{1}{c^2} \frac{\partial^2 \vec{B}}{\partial t^2} = -\mu_0 \nabla \times \vec{J}$$

} \Rightarrow complicated and have a fundamental drawback their solutions may not necessarily solve the original ME!

The way out \Rightarrow potentials

$$\vec{E} = -\nabla \phi - \frac{\partial \vec{A}}{\partial t}; \quad \vec{B} = \nabla \times \vec{A}$$

Gauge freedom: Let's take Lorenz gauge

$$\nabla \cdot \vec{A}_L + \frac{1}{c^2} \frac{\partial \phi_L}{\partial t} = 0$$

gives $\left\{ \begin{array}{l} \nabla^2 \phi_L - \frac{1}{c^2} \frac{\partial^2 \phi_L}{\partial t^2} = -\rho/\epsilon_0 \\ \nabla^2 \vec{A}_L - \frac{1}{c^2} \frac{\partial^2 \vec{A}_L}{\partial t^2} = -\mu_0 \vec{J} \end{array} \right\}$ much simpler source term.

Application 20.1 The fields of a point charge in Uniform Motion I

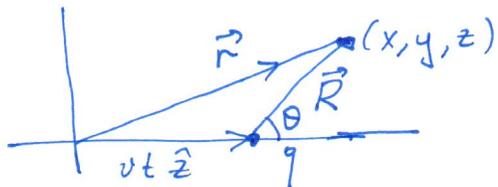
The original Lorenz solution for the field produced by a charge moving with a constant velocity $\vec{v} = v \hat{z}$

The goal is to solve for the potentials ϕ_L, \vec{A}_L

$$\rho(\vec{r}, t) = q \delta(x) \delta(y) \delta(z - vt), \quad \vec{j}(\vec{r}, t) = \vec{v} \rho(\vec{r}, t) \Rightarrow \text{sources} \quad (2)$$

Look for the solution in the form

$$\varphi(\vec{r}, t) = \varphi(x, y, \xi), \quad A(x, y, \xi) \hat{z}, \quad \xi \equiv z - vt$$



Set $\beta = v/c$, and note $\frac{\partial^2 \varphi}{\partial z^2} - \frac{\partial^2 \varphi}{\partial t^2} \frac{1}{c^2} = \left(\frac{\partial}{\partial z} - \frac{1}{c} \frac{\partial}{\partial t} \right) \cdot \left(\frac{\partial}{\partial z} + \frac{1}{c} \frac{\partial}{\partial t} \right)$

$$\left(\frac{\partial}{\partial z} \pm \frac{1}{c} \frac{\partial}{\partial t} \right) f(z - vt) = \left(1 \mp \frac{v}{c} \right) \frac{\partial f}{\partial \xi} = (1 \mp \beta) \frac{\partial f}{\partial \xi}$$

therefore the inhom. wave equation on the potential φ takes the form

$$\frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2} + (1 - \beta^2) \frac{\partial^2 \varphi}{\partial \xi^2} = -\frac{q}{\epsilon} \delta(x) \delta(y) \delta(\xi) \quad (*)$$

Introduce $\gamma^2 = \frac{1}{1 - \beta^2}$, and change variables to $z' = \gamma \xi = \gamma(z - vt)$

$$(1 - \beta^2) \frac{\partial^2 \varphi}{\partial \xi^2} = (1 - \beta^2) \gamma^2 \frac{\partial^2 \varphi}{\partial z'^2} = \frac{\partial^2 \varphi}{\partial z'^2} \quad \left. \begin{array}{l} \text{use it to} \\ \text{rewrite } (*) \end{array} \right\}$$

$$\delta(\xi) = \delta(\gamma^{-1} z') = \gamma \delta(z')$$

$$\frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2} + \frac{\partial^2 \varphi}{\partial z'^2} = -\frac{q \gamma}{\epsilon} \delta(x) \delta(y) \delta(z') \Rightarrow \text{Coulomb problem}$$

$$\varphi = \frac{1}{4\pi\epsilon_0} \frac{\gamma q}{\sqrt{x^2 + y^2 + \gamma^2 (z - vt)^2}}$$

$$\vec{A} = \frac{\mu_0}{4\pi} \frac{\gamma q v}{\sqrt{x^2 + y^2 + \gamma^2 (z - vt)^2}} \hat{z}$$

← obtained in the same way as the result for φ .

Let's obtain the electric field.

$$\vec{E} = -\nabla\phi - \frac{\partial \vec{A}}{\partial t}; \text{ the spatial derivative of } \phi \text{ give}$$

$$-\nabla\phi = \frac{\gamma q}{4\pi\epsilon_0} \frac{x\hat{x} + y\hat{y} + \gamma^2(z-vt)\hat{z}}{(x^2 + y^2 + \gamma^2(z-vt)^2)^{3/2}}$$

$$-\frac{\partial \vec{A}}{\partial t} = \mu_0 \frac{\gamma q}{4\pi} \frac{v^2 \gamma^2 \hat{z}}{(x^2 + y^2 + \gamma^2(z-vt)^2)^{3/2}}$$

the \hat{z} -component $\propto \frac{\gamma q}{4\pi\epsilon_0} \left[\gamma^2(z-vt)\hat{z} + \frac{1}{c^2} v^2 \gamma^2(z-vt)\hat{z} \right]$

$$\gamma^2(1 - \beta^2) = 1 \Rightarrow \propto \frac{\gamma q}{4\pi\epsilon_0} (z-vt)\hat{z}$$

in summary

$$\vec{E} = \frac{\gamma q}{4\pi\epsilon_0} \frac{x\hat{x} + y\hat{y} + (z-vt)\hat{z}}{(x^2 + y^2 + \gamma^2(z-vt)^2)^{3/2}}$$

Let's discuss the physical meaning of this result.

Define $\vec{R} = x\hat{x} + y\hat{y} + (z-vt)\hat{z}$ (see Fig.)

$$\sin^2\theta = \frac{x^2 + y^2}{x^2 + y^2 + (z-vt)^2}; \vec{E}(\vec{r}, t) \parallel \vec{R}$$

Clearly \vec{E} is along the vector \vec{R} . Now let's fix the magnitude of \vec{E} .

$$|\vec{E}|^2 \left(\frac{4\pi\epsilon_0}{q}\right)^2 = \gamma^2 \frac{R^2}{(x^2 + y^2 + \gamma^2(z-vt)^2)^3}$$

Let's do some algebra to put it in a more transparent form (4)

$$\begin{aligned} \gamma^2 \frac{R^2}{(x^2 + y^2 + \gamma^2(z-vt)^2)^3} &= \frac{\gamma^2}{\gamma^6} \frac{R^2}{[(x^2 + y^2)\gamma^{-2} + (z-vt)^2]^3} = \\ &= \frac{R^2}{\gamma^4} \frac{1}{[(x^2 + y^2)(1-\beta^2) + (z-vt)^2]^3} = \frac{R^2(1-\beta^2)^2}{[x^2 + y^2 + (z-vt)^2 - \beta^2(x^2 + y^2)]^3} = \\ &= \frac{R^2(1-\beta^2)^2}{[x^2 + y^2 + (z-vt)^2]^3} \left[1 - \beta^2 \frac{x^2 + y^2}{x^2 + y^2 + (z-vt)^2} \right]^3 = \frac{(1-\beta^2)^2}{R^4(1-\beta^2 \sin^2 \theta)^3} \end{aligned}$$

As a result

$$\vec{E} = \frac{q}{4\pi\epsilon_0} \hat{R} \frac{1-\beta^2}{R^2(1-\beta^2 \sin^2 \theta)^{3/2}}$$

is the electric field of a moving charge.

In a non-relativistic limit $\beta \ll 1 \Rightarrow \vec{E} \approx \frac{q}{4\pi\epsilon_0} \frac{\hat{R}}{R^2}$

Recall $\vec{R} = x\hat{x} + y\hat{y} + (z-vt)\hat{z}$ as a result it is just the static field translated in time with the charge.

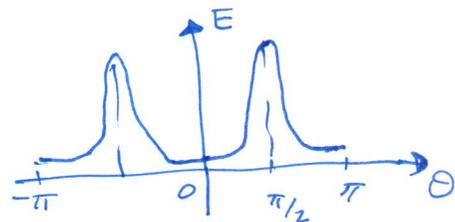
relativistic limit $\beta \lesssim 1$

$$\theta = 0 \Rightarrow \vec{E}_{\theta=0} = \frac{q}{4\pi\epsilon_0} \hat{z} \frac{1-\beta^2}{R^2}$$

$$E_{\theta=0} \ll \frac{q}{4\pi\epsilon_0} \frac{1}{R^2} \ll E_{\theta=\pi/2}$$

$$\theta = \pi/2 \Rightarrow \vec{E}_{\theta=\pi/2} = \frac{q}{4\pi\epsilon_0} \hat{R} \frac{1}{R^2(1-\beta^2)^{1/2}}$$

↳ this gives the maximum since $\sin \theta$ is at maximum



The total flux of electric field is fixed by the Gauss law

$$\oint \vec{E} \cdot \hat{R} dS = \frac{q}{4\pi\epsilon_0} \int_0^\pi d\theta \int_0^{2\pi} d\phi \frac{(1-\beta^2) \sin\theta}{(1-\beta^2 \sin^2\theta)^{3/2}} = \int_{x=\cos\theta}$$

↳ sphere of radius R

$$= \frac{q}{2\epsilon_0} \int_{-1}^1 dx \frac{(1-\beta^2)}{(1-\beta^2 + \beta^2 x)^{3/2}} = \frac{q}{\epsilon_0}$$

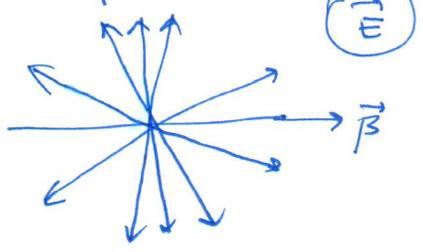
this holds regardless of β .

For $R \neq 0 \quad \nabla \cdot \vec{E} = \frac{1}{R^2} \frac{\partial}{\partial R} (R^2 E_R) = 0 \quad (E_\phi = E_\theta = 0)$

So the above result holds for any surface enclosing the charge.

For the surface not enclosing the charge the flux is 0.

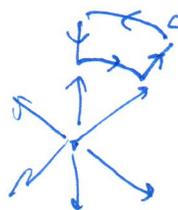
The picture at a fixed time



Lorentz "contraction"

~~The magnetic field can be shown to be $\vec{B} = \frac{\vec{v}}{c^2} \times \vec{E}(\vec{r}, t)$~~

in the non-relativistic approximation Faraday's law is ^{violated}

$v \ll c$  $\oint \vec{E} \cdot d\vec{l} = 0$ (the same as in electrostatics)

$\Rightarrow \nabla \times \vec{E} = 0 \Rightarrow \frac{\partial \vec{B}}{\partial t} = 0 \rightarrow$ is not the case!

Let's compute the magnetic field of a moving charge explicitly.

