

Appendix B

(1)

Normalization of solutions of the wave equation with the localized source.

Let's consider Eq. (21) of the notes

$$\frac{1}{R} \frac{d^2}{dR^2} (R G_k(R)) + k^2 G_k(R) = -4\pi \delta(\vec{R}) \quad (21 \text{ in notes})$$

Divide it by k^3

$$\frac{1}{k^3} \frac{1}{R} \frac{d^2}{dR^2} (R G_k(R)) + \frac{G_k(R)}{k} = -4\pi \delta(\vec{R}) \frac{1}{k^3}$$

Note $k^{-3} \delta(\vec{R}) = \delta(k\vec{R})$

Define $f_k(R) = k^{-1} G_k(R)$, and introduce the dimensionless variable $\vec{z} = k\vec{R}$

$$\frac{1}{z} \frac{d^2}{dz^2} (z f_k(z)) + f_k(z) = -4\pi \delta(\vec{z})$$

$$\Rightarrow \frac{d^2}{dz^2} (z f_k(z)) + (z f_k(z)) = 0 \quad \text{for } \underline{z \neq 0}$$

$$\Rightarrow f_k(z) = A_k \frac{e^{+i\vec{z}}}{z} + B_k \frac{e^{-i\vec{z}}}{z} \quad (*)$$

To fix A_k, B_k Note that the original equation takes the form

$$\vec{\nabla}_z^2 f_k + f_k = -4\pi \delta(\vec{z})$$

(**)

integrate (**) over the sphere of radius ϵ , and take $\epsilon \rightarrow 0$

$$\lim_{\epsilon \rightarrow 0} \left[\int_{V_\epsilon} d^3 \vec{z} \nabla_{\vec{z}}^2 f_k(\vec{z}) + \int_{V_\epsilon} d^3 \vec{z} f_k(\vec{z}) \right] = -4\pi$$

from (*) , $\lim_{\epsilon \rightarrow 0} \int_{V_\epsilon} d^3 \vec{z} f_k(\vec{z}) = 0$

Consider behavior of $f_k(\vec{z})$ at $\vec{z} \rightarrow 0$

$$f_k(\vec{z}) = A_k \frac{e^{i\vec{z}}}{z} + B_k \frac{e^{-i\vec{z}}}{z} \approx \frac{A_k + B_k}{z} \text{ at } z \ll 1$$

So as $\epsilon \ll 1$ we expect $f_k(\vec{z})$ to be almost identical with $(A_k + B_k)/z$, and become identical in the limit of $\epsilon \rightarrow 0$. We see then, that

$$\lim_{\epsilon \rightarrow 0} \left[\int_{V_\epsilon} d^3 \vec{z} \nabla_{\vec{z}}^2 f_k(\vec{z}) - \int_{V_\epsilon} d^3 \vec{z} \nabla_{\vec{z}}^2 \left(\frac{A_k + B_k}{z} \right) \right] = 0 \tag{***}$$

indeed as $z \ll 1$ $\nabla_{\vec{z}}^2 f(\vec{z}) \sim (A_k + B_k)/z^3$ and the same is true for $\nabla_{\vec{z}}^2 [(A_k + B_k)/z] \sim \frac{A_k + B_k}{z^3}$. As a result the integrand in (***) scales as $\sim (A_k + B_k) z^{-2}$ and is integrable. As a result we must have

$$\lim_{\epsilon \rightarrow 0} \int_{V_\epsilon} d^3 \vec{z} \nabla_{\vec{z}}^2 \left[\frac{A_k + B_k}{z} \right] = -4\pi$$

But since $\nabla_{\vec{z}}^2 \left(\frac{1}{z} \right) = -4\pi \delta(\vec{z})$

we obtain $A_k + B_k = 1$ for all k .