Emergence and Control of Multiphase Nonlinear Waves by Synchronization

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Large amplitude multiphase solutions of the periodic Korteweg–de Vries equation are excited and controlled by a small forcing. The approach uses passage through an ensemble of resonances and subsequent multiphase self-locking of the system with eikonal-type perturbations. The synchronization of each phase in the Korteweg–de Vries wave is robust, provided the corresponding driving amplitude exceeds a threshold.

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Waves in fluids and plasmas are frequently described by high order nonlinear partial differential equations having different classes of solutions. Suppose, one desires to generate (experimentally or in simulations) a waveform u of a particular class in the evolutionary system governed by $u_t + P(u) = 0$ (P being a nonlinear spatial differential operator). The standard procedure for achieving this goal requires knowledge and accurate realization of some nontrivial, frequently unknown *a priori*, initial conditions. Alternatively, in searching for a more realizable approach, we can start from simple (even vanishing) initial conditions, but consider a perturbed problem $u_t + P(u) = \varepsilon f$, where $\varepsilon \ll 1$ and f is a function on space-time. Then, we question the existence of a *simple* f, such that u in the perturbed system arrives at the desired nontrivial solution in the process of evolution. We shall show in this Letter that the answer to this question is positive for the driven (externally perturbed) Korteweg–de Vries (KdV) problem

$$u_t + 6uu_x + u_{xxx} = \varepsilon f(x, t). \tag{1}$$

The KdV equation ($\varepsilon = 0$) is one of the most important equations of nonlinear physics and describes many physical applications [1]. It has a variety of exact solutions, including traveling wave trains $u(x, t) = u(\theta)$, $\theta = kx - \omega t$, and the famous solitary waves. If one adds periodic boundary condition $u(x + L, t) = u(x, t)$, a nontrivial class of multiphase KdV solutions exists of form [2] $u(x, t) = u(\mathbf{\Theta})$, where $\mathbf{\Theta} = \{\theta_1, \ldots, \theta_N\}$ and each phase $\theta_n = \kappa_n x - \nu_n t$ has constant ν_n and κ_n (a multiple of $k_0 = 2\pi/L$ due to periodicity). The initial value problem for the periodic KdV equation is mathematically complex and realization of a given multiphase solution requires very special initial conditions. Consequently, these solutions are almost entirely dealt with on an advanced mathematical level [2,3] or via numerical simulations [4]. We shall show that adding a simple weak forcing and starting from zero, one can conveniently excite and control multiphase KdV waves. The goal is achieved by using a superposition $\varepsilon f(x, t) = \sum \varepsilon_n \sin\varphi_n$ of eikonal-type waves, where the phases $\varphi_n(x, t) = nk_0 x - \int \omega_n(t)dt$ have slowly varying frequencies $\omega_n(t)$, all passing, at say $t = 0$, the linear resonances [i.e., $-(nk_0)^3$] of the unperturbed problem. The number of phases (or open gaps in the main spectrum, see below) in the excited wave corresponds to the number of terms in the drive, while the frequencies and amplitudes of the emerging waveform are controlled by local values of the chirped frequencies. Excitation of a *single* phase KdV wave by adiabatic synchronization was studied in Ref. [5]. Later the theory was generalized to other extended systems [6,7] via Whitham's averaged variational principle [8]. Recent applications of similar ideas were reported in plasmas [9], fluid dynamics [10], and early evolution of the solar system [11]. Here, for the first time, we apply an adiabatic synchronization approach to controlling *multiphase* nonlinear waves.

We proceed by presenting results of numerical simulations illustrating our ideas. Figure 1 shows emergence of a 3-phase solution of driven periodic KdV Eq. (1). We used zero initial conditions and three driving components of amplitudes $\varepsilon_n = 0.09, 0.14, 0.11$, wave vectors $k_n = nk_0$

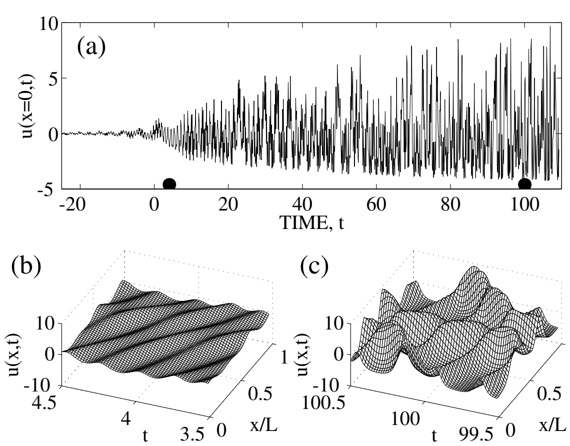

FIG. 1. Emergence of a 3-phase waveform u in configuration space. (a) $u(x = 0, t)$; (b),(c) $u(x, t)$ in two time windows $3.5 < t < 4.5$ and $99.5 < t < 100.5$ [indicated by full circles on the t axis in (a)].

($n = 1, 2, 3$) ($k_0 = 1$ in all our examples), and linearly chirped frequencies $\omega_n(t) = -(nk_0)^3 + \alpha_n t$, $\alpha_n = 0.08, 0.13, 0.10$. Figure 1(a) illustrates the evolution of $u(x = 0, t)$, while Figs. 1(b) and 1(c) show the solution in two narrow time windows of duration $\Delta t = 1.0$, centered around $t = 4.0$, and 100.0 [full circles on the t axis in Fig. 1(a)]. The excited waveforms are rather complex. Nevertheless, one observes the growth of the amplitude of oscillations far into the nonlinear stage, despite the smallness of the driving amplitudes ε_n. For diagnostics of these numerical results, we use spectral tools from the theory of the unperturbed, periodic KdV equation. N-phase solutions of this equation can be written as [2]

$$u(x, t) = 2 \sum_{n=1}^{N} \mu_n(x, t) - \sum_{m=1}^{2N+1} E_m, \tag{2}$$

where *all* objects are defined in terms of *three* spectral problems of the associated stationary Schrödinger equation with periodic potential $V(x) = -u(x, t)$, i.e.,

$$\psi_{xx} + [E + u(x, t)]\psi = 0, \tag{3}$$

where t is a parameter. The $2N + 1$ values E_m (the main spectrum) are the eigenvalues of (3), corresponding to eigenfunctions $\psi_\pm(x, t)$ having *periodic* or *antiperiodic* boundary conditions [$\psi_+(x + L, t) = \psi_+(x, t)$, and $\psi_-(x + L, t) = -\psi_-(x, t)$]. Importantly, the main spectrum of the unperturbed N-phase wave is *time independent* [3] and, therefore, fully defined by the initial condition, $u(x, 0)$. If E_m's are ordered by size, then the intervals $[E_{2n}, E_{2n+1}]$, $n = 1, 2, \ldots, N$ form gaps in the main spectrum. The N functions μ_n in Eq. (2) are multiphase objects $\mu_n(x, t) = \mu_n(\mathbf{\Theta}, \mathbf{E})$ ($\mathbf{E} = \{E_1, \ldots, E_{2N+1}\}$) and comprise another set of eigenvalues of Eq. (3) (the auxiliary spectrum), but their corresponding eigenfunctions ψ_0 have *zero* boundary conditions [$\psi_0(x + L, t) = \psi_0(x, t) = 0$]. Each $\mu_n(x, t)$ oscillates (in space/time) in one of the gaps, $E_{2n+1} \geq \mu_n(x, t) \geq E_{2n}$, and initial $\mu_n(x, 0)$ is sufficient for knowing μ_n at later times [3].

At this stage, we use the spectral machinery described above in studying properties of our *driven* solution. We again view t as a parameter, substitute our numerical waveform into Eq. (3), calculate the main spectrum $\mathbf{E}$ and show the evolution of $\mathbf{E}$ in Fig. 2(a). In contrast to Fig. 1, the spectral data in Fig. 2 reveal simple structure and evolution, i.e., the opening of three gaps at linear resonances, followed by the adiabatic increase of the width of each gap. A part of the evolution of the auxiliary spectrum μ_2 at $x = 0$ in the second gap is also shown in Fig. 2(a). Given the main spectrum $\mathbf{E}$, the theory allows one to find N frequencies ν_n [3] of the *imaginary* solution of the *unperturbed* KdV equation associated with our driven waveform. In other words, we view the driven solution, at given t, as the initial condition for an unperturbed wave, which approximates well our solution locally, and calculate frequencies ν_n of this imaginary wave. The dotted lines in Fig. 2(b) show the evolution of ν_n, while the chirped driving frequencies are represented by straight lines. One observes that starting $t \approx 0$, all three ν_n, on average, follow the corresponding driving frequencies, indicating a continuing phase locking (resonance) in the driven system. In addition to this averaged evolution, we observe oscillatory modulations of $\mathbf{E}$ and ν_n. These oscillations are characteristic signatures of the persistent phase locking (see theory below) and evolve on $O(\varepsilon^{1/2})$ time scale. Thus, beyond the linear resonance, each driving phase locks to one of the phases θ_n of the excited KdV wave, while, locally, the waveform corresponds to some solution of the unperturbed KdVequation, satisfying resonance conditions $\kappa_n = nk_0$ and $\nu_n(t) \approx \omega_n(t)$, $n = 1, 2, 3$. Since ν_n are functions of $\mathbf{E}$, the excited wave must adjust its main spectrum to stay in resonance and, therefore, the system modifies the open gaps' boundaries, affecting the amplitudes of μ oscillations as well. This means a full control of the excited wave by chirping the frequencies of the weak forcing.

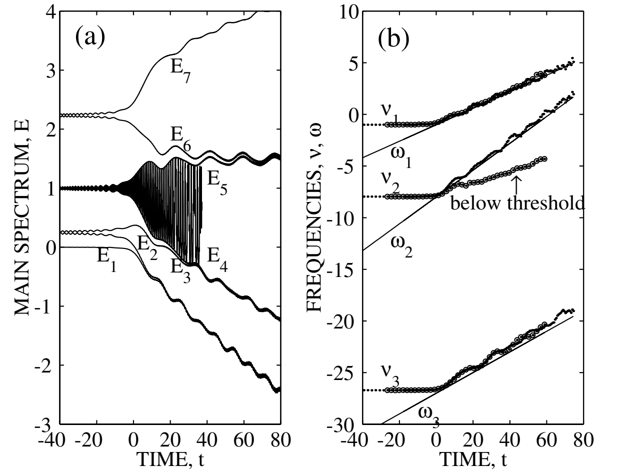

FIG. 2. The spectral analysis of the driven solution. (a) The opening of three main spectrum gaps $[E_{2n}, E_{2n+1}]$ by adiabatic synchronization (oscillations of the auxiliary spectrum μ_2 are shown in the gap $[E_4, E_5]$); (b) the frequencies ν_n of the excited wave (dots) and chirped driving frequencies ω_n (straight lines). The circles illustrate escape from resonance for ε_2 below the threshold.

The *passage through* linear resonances is an important stage for entering synchronized nonlinear evolution. Indeed, our simulations show that the phase locking is established during this early stage and sustained at later times, provided each driving amplitude exceeds a threshold. The thresholds $(\varepsilon_n)_{\rm cr}$ scale with the chirp rates as $\alpha_n^{3/4}$ (see below). We illustrate this phenomenon in Fig. 2(b), where circles show the results obtained by using all α_n and ε_n the same as before, but $\varepsilon_2 = 0.096$, i.e., 7% below the threshold value of $(\varepsilon_2)_{\rm cr} = 0.103$. One observes the departure of the second phase frequency from resonance beyond $t = 0$ in this case, i.e., the loss of phase locking with the second driving component. Two other phases remain locked to the corresponding drives, but *independent* control of *all* 3 degrees of freedom becomes impossible.

Now, we proceed to the theoretical interpretation of our results. First, we consider the initial multiphase locking stage. Because of zero initial conditions and weak driving, the initial wave excitation can be treated by a weakly nonlinear theory. We expect emergence of the solution of form $u(x, t) \approx \mathrm{Re} \sum s_n^m(t) \exp\{im[\kappa_n x - \phi_n(t)]\}$, where n denotes an *independent,* weakly nonlinear mode emerging due to the nth driving component, while m describes different order harmonics of mode n. In treating elementary excitations of such modes, we use single-phase theory [6]. Then real amplitudes $S_n = |s_n^1(t)|$ and phase mismatches $\Phi_n \equiv \arg(s_n^1)$ are governed by $dS_n/dt = -\varepsilon_n \sin\Phi_n$ and $d\Phi_n/dt = \alpha_n t - (3/2\kappa_n)S_n^2 - (\varepsilon_n/S_n)\cos\Phi_n$. We introduce new rescaled time $\tau = \alpha^{1/2} t$, amplitude $S_n' = \alpha_n^{-1/4}(3/2\kappa_n)^{1/2} S_n$, and driving parameter $\varepsilon' = \varepsilon_n \alpha_n^{-3/4}(3/2\kappa_n)^{1/2}$, yielding a single parameter system

$$\frac{dS_n'}{d\tau} = -\varepsilon' \sin\Phi_n, \qquad \frac{d\Phi_n}{d\tau} = \tau - S_n'^2 - \frac{\varepsilon'}{S_n'}\cos\Phi_n. \tag{4}$$

Numerical investigation of this system for $S_n' \to 0$ at $\tau \to -\infty$ shows that the asymptotic character of the solution at large positive τ differs depending on whether ε' is below or above a critical value of $\varepsilon'_{\mathrm{cr}} = 0.411$. For $\varepsilon' < \varepsilon'_{\mathrm{cr}}$, the amplitude S_n' reaches a constant at large τ, while the phase mismatch grows as $\Phi_n = \tau^2/2$. If $\varepsilon' > \varepsilon'_{\mathrm{cr}}$, in contrast, the amplitude grows as $S_n' = \tau^{1/2}$, while $|\Phi_n - \pi|$, (mod2π) remains bounded and small, indicating phase locking in the system. By returning to the original parameters, one obtains synchronization with the nth driving component in our driven KdV system, provided $\varepsilon_n > (\varepsilon_n)_{\mathrm{cr}} = 0.335\kappa_n^{1/2}\alpha_n^{3/4}$. A similar threshold phenomenon in passage through resonance was first studied in plasma applications [9].

Next, we discuss fully nonlinear evolution of the multiphase locked waves. We return to the spectral problem (3) with periodic potential $V = -u(x, t)$ as given by the solution of our driven KdV equation. Let E_γ and $\psi_\gamma(x, t)$ be one of the components of the main spectrum and the corresponding eigenfunction of this problem. Since $\mathbf{E}$ is fully determined by the set of frequencies and wave vectors $\{\nu_n, \kappa_n\}$ [3], while $\kappa_n = \mathrm{const}$ and $\nu_n(t) \approx \omega_n(t)$ in the phase locked state, one expects self-adjustment of $\mathbf{E}$ to sustain the resonance, i.e., E_γ becomes a slow function of time. To describe this slow evolution, we express $u(x, t)$ via E_γ and ψ_γ by using Eq. (3) and substitute the result into Eq. (1), yielding

$$E_{\gamma t}\psi_\gamma^2 + (\psi_\gamma Q_x - \psi_{\gamma x} Q)_x = -\varepsilon f(x, t)\psi_\gamma^2, \tag{5}$$

where $Q = \psi_{\gamma t} - \psi_{\gamma xxx} - 3(E_\gamma - u)\psi_{\gamma x}$. Then, upon averaging over the spatial period and using normalization $\langle \psi_\gamma^2 \rangle = 1$, we have [12]

$$dE_\gamma/dt = -\varepsilon\langle f(x, t)\psi_\gamma^2\rangle. \tag{6}$$

Importantly, only N out of $2N + 1$ components E_γ in Eq. (6) (say for *even* γ's) are independent. Indeed, conditions $\kappa_n(\mathbf{E}) = nk_0$ impose N relations between E_γ's [3], while an additional relation is obtained, by observing that our driven system (1) conserves the space average $\langle u(x, t)\rangle$. Next, we adopt multiscale representation [8,12,13], $\psi_\gamma^2 = F_\gamma(\mathbf{\Theta}, \mathbf{E}) + O(\varepsilon)$, $\gamma = 2n$. Here, the functional form of F_γ corresponds to an unperturbed KdV solution, therefore, F_γ is viewed as 2π periodic in all *fast* phase variables θ_n, but also evolves on the *slow* time scale via slowly varying main spectrum $\mathbf{E}(t)$ and frequencies $\overline{\nu}_n = -\partial\theta_n/\partial t = \nu_n[\mathbf{E}(t)] + O(\varepsilon)$. The $O(\varepsilon)$ addition in $\overline{\nu}_n$ represents the frequency shift due to the perturbation. We expand ψ_γ^2 in Eq. (6) in Fourier series in phase variables $\psi_\gamma^2 \approx \sum_{m_1, m_2, \ldots} b_{m_1, m_2, \ldots}^\gamma(\mathbf{E}) \exp[i(m_1\theta_1 + m_2\theta_2 + \cdots)]$. Then, by averaging Eq. (6) over the fast angles θ_n,

$$dE_\gamma/dt = \sum_{n=1}^{N} \varepsilon_n B_n^\gamma \sin\Phi_n^\gamma. \tag{7}$$

Here, the phase mismatch $\Phi_n^\gamma(t) \equiv \theta_n - [\kappa_n x - \int \omega_n(t)dt] + \sigma_n^\gamma$ is viewed as a slow function of time (the phase locking assumption), and we use the notation $b_{0,\ldots,m_n=1,0,\ldots}^\gamma = B_n^\gamma \exp(i\sigma_n^\gamma)$. We conjecture that σ_n^γ is independent of γ and omit this index in the phase mismatches in the following. Finally, by differentiation,

$$d\Phi_n/dt = \omega_n(t) - \nu_n[\mathbf{E}(t)] + O(\varepsilon), \qquad n = 1, \ldots, N. \tag{8}$$

The last two equations make up a complete set describing adiabatic synchronization in the system. Note that these developments show that one could use a much broader class of driving functions for achieving similar results. Indeed, let $f(x, t)$ be a space-time oscillation of form $F(x, \zeta_1, \zeta_2, \ldots, \zeta_N)$, which is L periodic in x, 2π periodic in all $\zeta_n = \zeta_n(t)$ with chirped associated frequencies $\omega_n(t) = -d\zeta_n/dt$. Then, expansion of F in the Fourier series in x and ζ_n and averaging in Eq. (6) again yields Eq. (7) with different coefficients. Also, unlike in the nonresonant modulational theory [8], where phases in slow evolution equations enter *indirectly* via frequencies and wave vectors, our slow resonant system involves phase mismatches *explicitly.* Equations (7) and (8) reduce to the small amplitude case (see above). Indeed, in this limit [14], $\psi_{2n}^2, \psi_{2n+1}^2 = 1 \pm \cos(\theta_n)$ and the amplitude of the nth weakly nonlinear mode is $S_n \approx E_{2n+1} - E_{2n}$. Then, Eq. (6) yields $dS_n/dt = -\varepsilon_n \sin\Phi_n$. Furthermore, in the weakly nonlinear stage, $\nu_n = \nu_n(S_n)$ in Eq. (8), while the $O(\varepsilon)$ term becomes important, because it scales as ε_n/S_n, while S_n is small initially. In the strongly nonlinear stage, in contrast, the $O(\varepsilon)$ term in Eq. (8) can be neglected, since the modulational equations predict slow oscillations of E_γ with amplitudes of $O(\varepsilon^{1/2})$ (see below). In analyzing modulational stability of a fully nonlinear synchronized stage, we seek solutions of Eqs. (7) and (8) of form $E_{2n} = \overline{E}_{2n} + \delta E_{2n}$ (we express the rest of the main spectrum in terms of E_γ with even γ only) and $\Phi_n = \overline{\Phi}_n + \delta\Phi_n$, where $\overline{E}_{2n}$ and $\overline{\Phi}_n \ll 1$ are smooth,

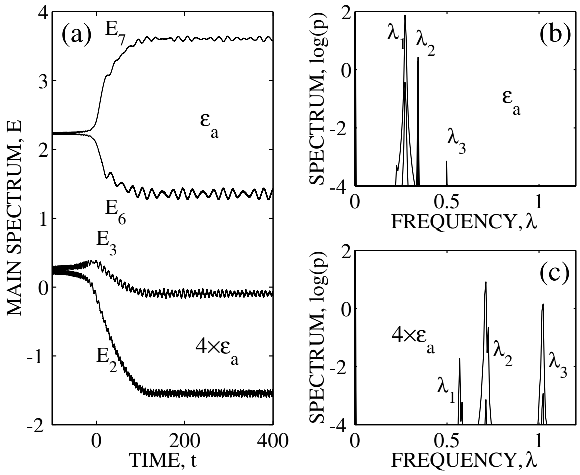

FIG. 3. Modulational stability of a synchronized 3-phase KdV state. All driving frequencies reach constant values beyond $t = 150$. (a) The evolution of $[E_2, E_3]$ and $[E_6, E_7]$ gaps for two sets of driving amplitudes $\varepsilon_{1,2,3} = \varepsilon_a = 0.067$ and $\varepsilon_{1,2,3} = \varepsilon_b = 4\varepsilon_a$, respectively. The power time-Fourier spectra of eigenvalues $E_{6,7}$ using drives ε_a (b), and of $E_{2,3}$ using drives ε_b (c). The characteristic frequencies λ double when ε_i quadruple.

slow averages, while δE_{2n}, $\delta\Phi_n$ are small and oscillating. The smooth $\overline{\mathbf{E}}(t)$ is given by $\underline{\omega_n}(t) - \nu_n[\overline{\mathbf{E}}(t)] = 0$ and can be used to find $\overline{\Phi}_n$ from $d\overline{E}_{2n}/dt = \sum_{m=1}^{N} \varepsilon_m \overline{B}_m^{2n} \sin\overline{\Phi}_m$. When the smooth objects are known, the oscillating components are governed by $d(\delta E_{2n})/dt = \sum_{m=1}^{N} \varepsilon_m \overline{B}_m^{2n} \delta\Phi_m$ and $d(\delta\Phi_m)/dt = -\sum_{l=1}^{N} (\partial\overline{\nu}_m/\partial\overline{E}_{2l})(\delta E_{2l})$. By fixing the coefficients in this linear problem locally and seeking solutions of form $\delta E_{2n}, \delta\Phi_n \sim \exp(-i\lambda t)$, we obtain the characteristic equation $\det[M - \lambda^2 I] = 0$, where $M_{nl} = \sum_{m=1}^{N} \varepsilon_m (\partial\overline{\nu}_m/\partial\overline{E}_{2l}) \overline{B}_m^{2n}$ and I is the identity matrix. The characteristic equation yields N real frequencies λ_n of $O(\varepsilon^{1/2})$ if the evolution is stable. Then all δE_{2n} scale as $O(\varepsilon^{1/2})$, justifying the neglect of the $O(\varepsilon)$ term in Eq. (8). In the *weakly* nonlinear evolution stage, above threshold, all λ_n are real, while our numerical examples show phase locking in the system and oscillating modulations of both $\mathbf{E}$ and the excited wave frequencies around slowly evolving averages.

In order to test the predictions of our modulational theory in more detail, we return to numerics. We consider excitation of a 3-phase solution by synchronization for the case when all driving frequencies approach values $\omega_{i0} = 3, -4, -23$ gradually at $t = 150$ and remain constant beyond this time. Some results of these calculations are presented in Fig. 3. We used parameters $\alpha_{1,2,3} = 0.05$ (at linear resonances) and two sets, $\varepsilon_{1,2,3} = \varepsilon_a = 0.067$ and $\varepsilon_b = 4\varepsilon_a$. We have found that, as predicted, the averaged evolution of the main spectra was nearly the same for both sets of ε_i. The modulations of E_i, in contrast, were different. Figure 3(a) shows the evolution of the $[E_2, E_3]$ gap (for set ε_b), and the $[E_6, E_7]$ gap (for set ε_a); the gap $[E_4, E_5]$ is not shown in the figure. We observe that the gap widths remain constant in average, as the driving frequencies stay unchanged beyond $t = 150$, and time Fourier analyze E_j of the resulting quasisteady synchronized state. The power spectra $p(\lambda)$ of the modulations for the two sets of ε_i are shown in Figs. 3(b) and 3(c). We see that there indeed exist *three* characteristic frequencies λ_i and that these frequencies double when one quadruples ε_i. The results illustrate stability of the excited synchronized KdV states, as well as the predicted $O(\varepsilon^{1/2})$ scaling of the characteristic modulation frequencies.

In conclusion, we have studied passage through multiple resonances and synchronization in the periodic KdV equation. The passage allows efficient excitation and control of N-phase waves by starting from trivial (zero) initial conditions and using a weak $[O(\varepsilon)]$ forcing. The thresholds for capture into resonance and small modulations of the main spectra, oscillating at $O(\varepsilon^{1/2})$ frequencies are the main signatures of synchronized KdV states. The simplicity of the forcing studied in this work may bridge between physics and mathematics in the field, making the generation and control of nontrivial multiphase waveforms experimentally realizable. Applications of a similar, multifrequency control, leading to the emergence of multiphase solutions in other fundamental nonlinear wave systems, seem to be an interesting direction for future research.

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