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MIGRATION TIMESCALE THRESHOLDS FOR RESONANT CAPTURE IN THE PLUTINO PROBLEM

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ABSTRACT

Dynamic autoresonance theory is applied to the problem of thresholds on migration timescales for capture into resonances in the planar-restricted three-body problem with $m_1 \gg m_2 \gg m_0$ and slowly migrating masses $m_{1,2}$. The thresholds are found analytically, scale as $(m_2/m_1)^{-4/3}$, and yield an order of magnitude longer timescales required for capture of m_0 into 2 : 1 outer resonance as compared with 3 : 2 and other resonances. The difference is due to the rotation of the primary mass m_1, affecting the 2 : 1 resonance only. This could explain the observed small abundance of Kuiper Belt objects in the 2 : 1 resonance and could define accurate bounds on the timescales involved in the early evolution of the solar system.

Subject headings: celestial mechanics, stellar dynamics — Kuiper Belt — planets and satellites: individual (Neptune) — solar system: formation

1. INTRODUCTION

It is widely accepted that Pluto's peculiar orbit, having large eccentricity and phase locking in 3 : 2 mean motion resonance with Neptune, is the result of Neptune's orbital passage through the resonance at a late stage of planetary accretion. According to this scenario, the passage led to the capturing of Pluto as well as many other objects in the Kuiper Belt into resonance (Malhotra 1993). Observations show that many Kuiper Belt objects (KBOs, so-called Plutinos) are in the 3 : 2 resonance with Neptune.[1] Early numerical simulations, supporting the resonant capture scenario (Malhotra 1995), also predicted a comparable number of KBOs trapped in the 2 : 1 resonance. However, very few such KBOs are observed. This contradiction (one of the major remaining questions of the resonant KBO capture theory; Jewitt & Luu 2000) was discussed in a recent work by Ida et al. (2000). This study, based on numerical simulations, suggested that capture into 2 : 1 resonance required much longer Neptune migration timescales as compared with those for 3 : 2 resonance. The simulations were supplemented by scenarios yielding sufficiently short migration times for preventing capture into the 2 : 1 resonance.

In the present work, we use dynamic autoresonance theory in studying the problem of timescale filtering in the Plutino problem *analytically*. The term *autoresonance* is used in describing the continuing phase locking in driven nonlinear systems with slow parameters. The theory of autoresonance was extended recently to applications in fluid dynamics (Friedland 1999), plasmas (Fajans, Gilson, & Friedland 1999), and nonlinear waves (Friedland & Shagalov 1998). One of the main predictions of this theory is the existence of a *sharp threshold* on the driving frequency sweep rate (the rate of variation of the angular frequency of Neptune in the Plutino problem) for capture into resonance. We shall find these thresholds for the KBO problem and show that the minimum Neptune migration timescales for capture into 3 : 2 or 2 : 1 resonances must be longer than 1.7 or 19.7 Myr, respectively. This large difference of timescales is due to the Sun's rotation around the center of mass, affecting 2 : 1 resonance only.

The following section introduces our *adiabatic*-restricted three-body model, while § 3 deals with the issue of thresholds, numerical illustrations, and conclusions.

[1] See data in B. G. Marsden's Web site (http://cfa-www.harvard.edu/cfa/ps/lists/TNOs.html).

2. ADIABATIC-RESTRICTED THREE-BODY PROBLEM

Our starting point is the planar three-body Hamiltonian

$$H = \frac{1}{2}\left(p_r^2 + \frac{p_\varphi^2}{r^2}\right) - \frac{1}{\rho_1} - \frac{q}{\rho_2}, \qquad (1)$$

describing a test particle of mass m_0 in the gravitational field of two dominant masses $m_{1,2}$ rotating around their center of mass on expanding, nearly circular orbits (see Fig. 1) unperturbed by the test particle. The distances $\rho_{1,2}$ in equation (1) are $\rho_1^2 = r^2 + r_1^2 + 2rr_1\cos(\varphi - \psi)$ and $\rho_2^2 = r^2 + r_2^2 - 2rr_2\cos(\varphi - \psi)$, where $m_1 r_1 = m_2 r_2$ and $\psi(t)$ is the rotation angle of $m_{1,2}$, such that the angular velocity $d\psi/dt = \omega(t)$ is a *slow* function of time, i.e., $\gamma \equiv \omega^{-2}A \ll 1$, where $A = |d\omega/dt|$. The inclusion of the slow variation of the frequency of $m_{1,2}$ is the main difference between the present model (referred to as the *adiabatic*-restricted three-body problem in the following) and the conventional *circular*-restricted three-body problem with fixed parameters. Another assumption of our theory is that the mass ratio $q = m_2/m_1$ is a small parameter ($q \ll 1$). An example of such a system is m_1, m_2, and m_0 representing the Sun, Neptune, and a KBO, while the slow time variation of ω is due to the migration of Neptune's orbit at an early stage of the evolution of the solar system. We shall consider the case in which the test particle starts on a circular orbit of radius $r = r_0$ and an angular frequency $\omega_0 \approx (Gm_1/r_0^3)^{1/2}$. Note that we replaced Gm_1 in equation (1) by unity, equivalent to using dimensionless time $t \to \omega_0 t$, dimensional radial coordinate $r \to r/r_0$ and normalizing the momenta $p_r \to p_r r_0/p_{\varphi 0}$, $p_\varphi \to p_\varphi/p_{\varphi 0}$ with respect to $p_{\varphi 0} = \omega_0 r_0^2$. Thus, our initial conditions are $r = 1$, $\varphi = \varphi_0$, $p_r = 0$, and $p_\varphi = 1$. Our goal is to study the capture of the test particle into an orbit-orbit resonance as the angular frequency of masses $m_{1,2}$ varies in time.

As a first step in studying the passage through resonance, we expand $1/\rho_1$ in equation (1) in powers of small parameter q and truncate the expansion at $O(q^1)$. This yields the approximate Hamiltonian $H \approx H_0(p_r, p_\varphi, r) + qV_1(r, \Theta)$, where $H_0 =$

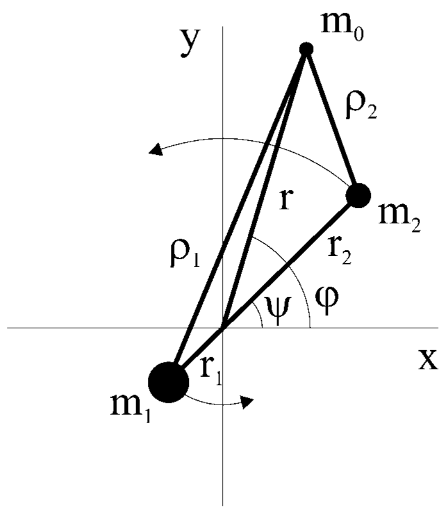

FIG. 1.—Geometry of the adiabatic-restricted three-body problem. The primary and secondary masses m_1 and m_2 move on expanding, spiraling orbits having a slowly varying local angular frequency $\omega(t) = d\psi/dt$.

$\frac{1}{2}\,(p_r^2 + p_\varphi^2/r^2) - 1/r$, and

$$r_2V_1 \equiv \alpha^2 \cos\Theta - \alpha(1 + \alpha^2 - 2\alpha\cos\Theta)^{-1/2}, \qquad (2)$$

with $\alpha \equiv r_2/r$ and $\Theta \equiv \varphi - \psi$. We shall focus on outer resonances; i.e., we assume that $\alpha < 1$. Note that the first term on the right-hand side in equation (2) is due to the rotation of the primary mass m_1, while the second term represents the interaction with the secondary mass m_2. One can also rewrite equation (2) as an expansion:

$$r_2V_1 = \sum_{j=0}^{\infty} f_j \cos(j\Theta), \qquad (3)$$

where $f_0 = -(\alpha/2)b_{1/2}^{(0)}(\alpha)$, $f_1 = [\alpha^2 - \alpha b_{1/2}^{(1)}(\alpha)]$, and $f_j = -\alpha b_{1/2}^{(j)}(\alpha)$ for $j > 1$, while $b_{1/2}^{(j)}$ are the Laplace coefficients. We see that the motion of m_1 affects the coefficient f_1 only in equation (3) and always yields a *reduction* of this coefficient. We shall see later that due to this effect (and since the term with f_1 in eq. [3] is responsible for the outer $2:1$ resonance), the minimum migration timescale for capture into this resonance increases considerably.

As the second preliminary step for analyzing the passage through resonance in our problem, we transform to canonical radial and azimuthal action-angle variables (J_r, Θ_r) and $(J_\varphi, \Theta_\varphi)$ associated with the unperturbed Hamiltonian H_0. The actions $J_{r,\varphi}$ are dimensionless and correspond to the normalization of the corresponding dimensional actions $I_{r,\varphi}$ with respect to $I_{\varphi 0} = \omega_0 r_0^2$; i.e., $J_{r,\varphi} \equiv I_{r,\varphi}/I_{\varphi 0}$. We shall see below that capture into resonance in our system is a weakly nonlinear effect in terms of J_r for the test particle starting on a circular orbit ($J_r = 0$). Consequently, we use a small J_r approximation. Then (see Appendix) the problem reduces to studying the following pair of evolution equations for $\Delta \equiv (2J_r)^{1/2}$ and phase mismatch $\Phi \equiv j\Theta_\varphi + \Theta_r - j\psi(t)$ associated with the $j+1:j$ resonance:

$$d\Delta/d\tau = \epsilon \sin\Phi, \qquad (4)$$

$$d\Phi/d\tau = \tau - \frac{p}{8}\Delta^2 + (\epsilon/\Delta)\cos\Phi. \qquad (5)$$

In deriving these equations in the Appendix, we assumed a linear frequency sweeping $\omega(t) = (j+1)/j - At$ through $j + 1:j$ resonance ($A = d\omega/dt$ being the sweep rate) and introduced a rescaled time $\tau \equiv (jA)^{1/2}t$. The nonlinearity and coupling parameters in equations (4) and (5) are $p = 12(j+1)^2 \times (jA)^{-1/2}$ and $\epsilon = q\eta_j(jA)^{-1/2}$, where η_j can be expressed via Laplace coefficients at the resonance. We shall also assume that $\varepsilon/p = (q\eta_j)/[12(j+1)^2] \ll 1$ in the following.

3. TIMESCALE THRESHOLD PHENOMENON

Now we proceed to the problem of thresholds. The variables $J \equiv \Delta^2$ and Φ described by equations (4) and (5) comprise a canonical pair, and the corresponding Hamiltonian is $H_{\rm eff} = \tau J - (p/16)\,J^2 + 2\epsilon J^{1/2}\cos\Phi$. Capture into resonance for Hamiltonians of this type was studied by Henrard (1982). However, this study did not cover the limit of initially small J, when equation (5) is singular. The capture associated with this singularity and the subsequent autoresonance in the system were considered in more recent theories (e.g., Friedland 1999). We shall not repeat the details of this analysis here but mention the two main conclusions of the theory:

Conclusion A.—If one starts at sufficiently large negative times τ and sufficiently small initial Δ, then, at some negative time (still prior the linear resonance point $\tau = 0$), the system phase locks with the drive, i.e., the phase mismatch $\Phi \to 0$ regardless initial Φ.

Conclusion B.—Later, as the system passes through the linear resonance, i.e., τ becomes positive, the phase locking $\Phi \approx 0$ continues, provided the driving parameter ϵ exceeds a threshold:

$$\epsilon > \epsilon_{\rm th} = (4/3)^{3/4}p^{-1/2}. \qquad (6)$$

Below this threshold, the phase locking discontinues. In the present case, where $\epsilon = q\eta_j(jA)^{-1/2}$ is given, equation (6) translates into the condition on A to be less than some critical value for capture into resonance:

$$A < A_j^{\rm th} = (3.93/j)[q(j+1)\eta_j]^{4/3}. \qquad (7)$$

Here the coefficients η_j are evaluated at the linear resonance, i.e., using $\alpha = [j/(j+1)]^{2/3}$. The theory leading to the threshold (eq. [6]) is simple, and we present it below for completeness.

Let us assume a continuing phase locking $\Phi \approx 0$ in the system as it passes the linear resonance point and, consequently, replace $\cos\Phi$ in equation (5) by unity. Then let us differentiate the resulting equation in time and substitute equation (4) for $d\Delta/d\tau$, yielding

$$d^2\Phi/d\tau^2 = 1 - \epsilon S \sin\Phi, \qquad (8)$$

where $S \equiv p\Delta/4 + \epsilon/\Delta^2$. Equation (8) describes a quasi-particle in the tilted cosine potential:

$$V_{\rm eff}(\Phi) = -\Phi - \epsilon S \cos \Phi, \quad (9)$$

where parameter S is a function of time via slowly varying Δ in S. The quasi-potential $V_{\rm eff}(\Phi)$ possesses minima only if the tilting is not too large, i.e., when $\epsilon S > 1$. Otherwise $V_{\rm eff}(\Phi)$ decreases monotonically. The existence of the potential minima is necessary for having trapped solutions, i.e., phase locking in our real problem, where Φ describes the phase mismatch between the driving perturbation and the test particle. On the other hand, S has a minimum $S_m = \frac{3}{4}\epsilon^{1/3}p^{2/3}$ at $\Delta = \Delta_m = 2(\epsilon/p)^{1/3}$. Substitution of S_m into $\epsilon S > 1$ yields the threshold condition (6). Finally, because of the assumed smallness of ϵ/p, Δ_m is small, justifying our weakly nonlinear treatment of the resonant trapping problem.

Now we use equation (7) to find the threshold timescales for capture into 2 : 1, 3 : 2, and 4 : 3 resonances ($j = 1$, 2, and 3). The coefficients η_j are evaluated numerically, yielding $\eta_{1,2,3} = 0.43$, 2.48, and 3.28. Then $A_j^{\rm th} = \kappa_j q^{4/3}$, where $\kappa_{1,2,3} = 3.2$, 28.5, and 40.5. Note that $A_1^{\rm th}$ is almost 10 times smaller than $A_2^{\rm th}$. This difference is caused by the inclusion of the primary mass m_1 rotation around the center of mass (e.g., the Sun's motion in the Plutino problem). If one neglects this rotation, one obtains $\eta_1 = 1.69$ and $\kappa_1 = 19.9$, i.e., much larger $A_1^{\rm th}$. Now we use $A_j^{\rm th}$ in calculating the minimum m_2 migration timescale (dimensional) for resonant capture of the test particle:

$$t_j^{\rm th} \equiv \frac{r_2}{(dr_2/dt)_{\rm th}} = \frac{3\omega}{2|d\omega/dt|_{\rm th}} = \frac{3(j+1)}{2j\kappa_j\omega_0}\, q^{-4/3}. \quad (10)$$

This expression applied to the Plutino problem ($q = m_2/m_1 = 5.13 \times 10^{-5}$, and, say, $\omega_0 = 0.025$ rad yr^{-1}) yields $t_{1,2,3}^{\rm th} = 19.7$, 1.7, and 1.0 Myr for capture into 2 : 1, 3 : 2, and 4 : 3 resonances, respectively. The dependence $t_{\rm mig} = 9.5q^{-4/3}$ yr for the migration timescale $r_2/(dr_2/dt)$, yielding *high* capture probability into 3 : 2 resonance and *low* capture probability into 2 : 1 resonance, was suggested by Ida et al. (2000) on the basis of simulations. In the Plutino case, this formula yields $t_{\rm mig} = 5$ Myr, in full agreement with the present theory, since $t_2^{\rm th} < t_{\rm mig} < t_1^{\rm th}$ (see our thresholds above). Earlier simulations (Malhotra 1995) used a progressively slowed down migration model. The shortest $t_{\rm mig}$ was 6.6 Myr, but after 2 Myr of migration in these runs, $t_{\rm mig}$ exceeded 21 Myr and continued to increase. Therefore, $t_{\rm mig}$ was longer than $t_{1,2}^{\rm th}$ at later times, and efficient capture was observed at both 3 : 2 and 2 : 1 resonances.

At this stage, we present numerical results illustrating our theory. The open circles in Figure 2 show the threshold migration timescales $t_{1,2}^{\rm th}$ for trapping into 2 : 1 and 3 : 2 resonances versus q, as found by integrating the full set of evolution equations of the adiabatic-restricted three-body problem. We also show

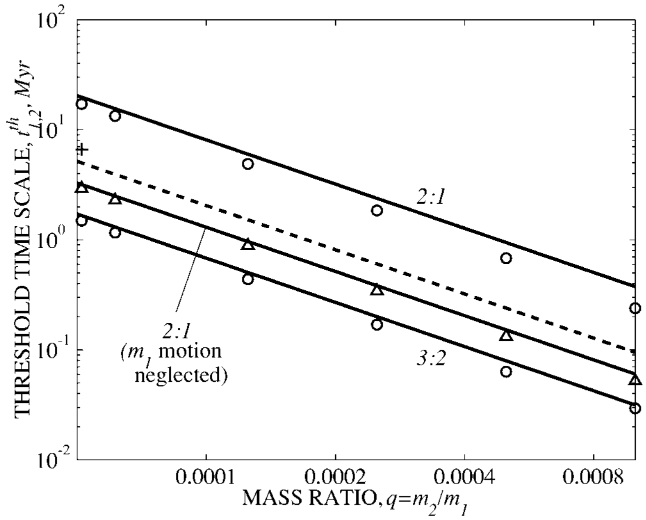

FIG. 2.—Threshold timescale $t_{1,2}^{\rm th}$ for capture into 2 : 1 and 3 : 2 resonances vs. $q = m_2/m_1$. *Open circles*: Results from exact equations of the adiabatic-restricted three-body problem. *Open triangles*: Same for the 2 : 1 resonance, but neglecting rotation of m_1. The solid lines represent the scalings (eq. [10]). The dashed line is $t_{\rm mig} = 9.5q^{-4/3}$ (Ida et al. 2000), while the plus sign is the shortest $t_{\rm mig}$ in earlier simulations (Malhotra 1995).

(Fig. 2, *open triangles*) the results of the same calculations for 2 : 1 resonance, but we neglect the rotation of the primary mass. One observes a factor of 6 decrease of $t_1^{\rm th}$ in this case, illustrating the main reason for a large difference between the timescales for trapping into 2 : 1 and 3 : 2 resonances. The solid lines in Figure 2 represent the theoretical scalings (eq. [10]) in all the cases. One can see that, generally, the theoretical curves are only 10%–20% higher than the numerical results, while the $q^{-4/3}$ scaling is in very good agreement with the calculations. Finally, the dashed line in the Figure 2 is $t_{\rm mig} = 9.5q^{-4/3}$ yr from Ida et al. (2000), representing their regime of high capture probability into 3 : 2 resonance and low capture probability into 2 : 1 resonance, while the plus sign in Figure 2 show the shortest $t_{\rm mig}$ in simulations by Malhotra (1995).

We have studied the capture into resonances in the adiabatic-restricted three-body problem. In conclusion, the theory yields analytic thresholds on the migration timescales and the order-of-magnitude difference in the thresholds for capture into 3 : 2 and 2 : 1 resonances, providing an explanation for the small observed abundance of KBOs in the 2 : 1 resonance. Inclusion of finite inclination, its excitation via frequency sweeping mechanism, and studying the associated migration rate thresholds comprise interesting extensions of the theory in the future.

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APPENDIX

REDUCED ACTION-ANGLE FORMULATION

Here we present the details of the reduction procedure yielding equations (4) and (5) for studying capture into resonance in the adiabatic-restricted three-body problem. We transform to dimensionless radial and azimuthal action-angle variables (J_r, Θ_r) and (J_φ, Θ_φ) associated with the unperturbed Hamiltonian H_0. It is well known that under the canonical transformation, H_0 becomes $H_0 = -\frac{1}{2}(J_r + J_\varphi)^{-2}$. The capture into resonance in our case is a weakly nonlinear effect in terms of J_r when starting on a circular orbit ($J_1 = 0$), so we shall use a small J_r approximation. Furthermore, since qV_1 in the Hamiltonian already involves small parameter

q, we shall approximate V_1 to first order in the amplitude of radial oscillations. Thus, we write $r = 1 + \delta$, $\delta \ll 1$ and expand in equation (3) to first order in δ:

$$V_1 \rightarrow \sum_{j=0}^{\infty} (C_j + \delta B_j) \cos(j\Theta), \tag{A1}$$

where C_j and B_j are $\alpha^{-1} f_j$ and $-\partial f_j/\partial\alpha$ evaluated at $r = 1$ (i.e., $\alpha = r_2$).

Next we express δ and φ in terms of the action-angle variables (using linear approximation for the radial oscillations):

$$\delta = (2J_r)^{1/2} \cos\Theta_r, \qquad \varphi = \Theta_\varphi + R(\Theta_r, J_r, J_\varphi), \tag{A2}$$

where $R(\Theta_r, J_r, J_\varphi)$ is associated with the generating function of the canonical transformation:

$$R = -\int^r (\partial p_r^*/\partial J_\varphi) dr'. \tag{A3}$$

Here $p_r^*(J_r, J_\varphi, r)$ is the solution of $H_0(J_r, J_\varphi) = \frac{1}{2}(p_r^2 + J_\varphi^2/r^2) - \frac{1}{r}$ for p_r (recall that $J_\varphi = p_\varphi$); i.e.,

$$p_r^*(J_r, J_\varphi, r) = \pm[2H_0 + 2/r - (J_\varphi/r)^2]^{1/2}. \tag{A4}$$

Thus,

$$R = \int (J_\varphi/r'^2 - \partial H_0/\partial J_\varphi) dt, \tag{A5}$$

with integration in time along the unperturbed motion. Since, to lowest order, $J_\varphi = 1$ and $\partial H_0/\partial J_{\varphi,r} = d\Theta_{\varphi,r}/dt = 1$, while $r = 1 + \delta$, we have $R \approx -2\int \delta\, dt = -2(2J_r)^{1/2} \sin\Theta_r$. This allows us to approximate $\cos(j\Theta) \approx \cos[j(\Theta_\varphi - \psi)] + 2(2J_r)^{1/2} j \times \sin\Theta_r \sin[j(\Theta_\varphi - \psi)]$ in equation (A1):

$$V_1 = V_{10} + (2J_r)^{1/2} \sum_{j=0}^{\infty} \{B_j \cos\Theta_r \cos[j(\Theta_\varphi - \psi)] + 2C_j j \sin\Theta_r \sin[j(\Theta_\varphi - \psi)]\},$$

where $V_{10} = \sum_{j=0}^{\infty} C_j \cos[j(\Theta_\varphi - \psi)]$. Finally, we leave one resonant term in V_1 and write the single resonance Hamiltonian

$$H_j = -\frac{1}{2}(J_r + J_\varphi)^{-2} + q(2J_r)^{1/2} \eta_j \cos[j(\Theta_\varphi - \psi) + \Theta_r], \tag{A6}$$

where $\eta_j = B_j/2 - jC_j$.

The Hamiltonian H_j yields

$$dJ_r/dt = q\eta_j (2J_r)^{1/2} \sin\Phi, \tag{A7}$$

$$dJ_\varphi/dt = jq\eta_j (2J_r)^{1/2} \sin\Phi, \tag{A8}$$

$$d\Phi/dt = (j+1)\Omega - j\omega + q\eta_j (2J_r)^{-1/2} \cos\Phi, \tag{A9}$$

where $\Phi \equiv j\Theta_\varphi + \Theta_r - j\psi(t)$ and $\Omega = (J_r + J_\varphi)^{-3}$ are the phase mismatch and the Keplerian frequency of the test particle, respectively. These evolution equations are the same as in the system with fixed parameters, but now $\omega(t)$ and η_j are slow functions of time. Nevertheless, equations (A7) and (A8) still yield the conservation law $J_\varphi - jJ_r = J_{\varphi 0} = 1$, or $J_\varphi = 1 + jJ_r$, which, to lowest order in J_r, gives

$$\Omega = [1 + (j+1)J_r]^{-3} \approx 1 - 3(j+1)J_r. \tag{A10}$$

Thus, there remain only two independent variables (say, J_r and Φ) in the problem. Finally, by assuming linear frequency sweeping $\omega(t) = (j+1)/j - At$ through $j+1:j$ resonance and introducing $\Delta \equiv (2J_r)^{1/2}$, $\tau \equiv (jA)^{1/2} t$, $p = 12(j+1)^2 (jA)^{-1/2}$, and $\epsilon = q\eta_j (jA)^{-1/2}$, one obtains the reduced system (eqs. [4] and [5]) introduced in § 2.

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